The Open University

MST121
Using Mathematics

Exercise Book C

Contents

The exercises in this booklet are intended to give further practice, should you require it, in handling the main mathematical ideas in each chapter of MST121, Block C. The exercises are ordered by chapter and section, and are numbered correspondingly: for example, Exercise 3.2 for Chapter C1 is the second exercise on Section 3 of that chapter.

Exercises for Chapter C1

Section 1

Exercise 1.1

For each of the following quadratic functions f, write down an expression for $f'(x)$, and find the gradient of the graph of $y = f(x)$ at the point given.

(a) $f(x) = x^2 - 7x \quad (9, 18)$

(b) $f(x) = 2 - 3x^2 \quad (2, -10)$

(c) $f(x) = 7x^2 - 12x - 19 \quad (1, -24)$

Exercise 1.2

The function f is given by $f(x) = 2x^2 - 5x + 1$.

(a) Write down an expression for $f'(x)$.

(b) Find the gradient at the point $(2, -1)$ on the graph of $y = f(x)$.

(c) Using a result from Chapter A2, find the equation of the line which is a tangent to the graph at the point $(2, -1)$.

(d) Find the point on the graph at which the gradient is -9.

Exercise 1.3

(a) Show that if $x \neq 0$ and $x + h \neq 0$, then

$$\frac{1}{(x+h)^2} - \frac{1}{x^2} = -\frac{h(2x+h)}{x^2(x+h)^2}.$$

(b) Use equation (1.3) of Chapter C1 Section 1 and the result of part (a) to differentiate the function

$$f(x) = \frac{1}{x^2} \quad (x \neq 0).$$

(c) Verify that the result of part (b) can also be obtained by applying result (1.4) of Chapter C1 Section 1.

Exercise 1.4

Differentiate each of the following functions.

(a) $f(x) = x^{17}$

(b) $f(x) = \dfrac{1}{x^{17}}$

(c) $f(x) = \dfrac{1}{\sqrt{x}}$

(d) $f(x) = x\sqrt{x}$

Exercise 1.5

A ball thrown vertically upwards has position $s = 15t - 5t^2$ metres above its starting point after t seconds.

(a) Find the velocity $v\,\mathrm{m\,s^{-1}}$ of the ball at time t.

(b) What is the velocity of the ball after 1 second?

(c) At what time does the ball have velocity zero?

(d) What is the height of the ball when its velocity is zero?

Section 2

Exercise 2.1

Differentiate each of the following polynomial functions.

(a) $f(x) = \frac{3}{2}x^{10} - 2x^6 + 3x$

(b) $f(x) = 7 + 4x^2 - 3x^4 + x^6$

Exercise 2.2

(a) What is the gradient of the graph of $y = \frac{1}{4}x^8 - \frac{1}{3}x^6 + \frac{1}{3}$ at the point $(2, 43)$?

(b) What is the gradient of the graph of $y = x^5 - 3x^3 - x^2 - 6$ at the point $(-1, -5)$?

Exercise 2.3

(a) Find the derivative of the function

$$f(x) = 3x^2 + 18x + 15.$$

(b) Show that

$$f'(-3) = 0;$$
$$f'(x) < 0 \text{ for } x \text{ in } (-\infty, -3);$$
$$f'(x) > 0 \text{ for } x \text{ in } (-3, \infty).$$

(c) Deduce from part (b) the intervals on which $f(x)$ is increasing or decreasing.

(d) Find $f(0)$, $f(-3)$ and the values of x for which $f(x) = 0$.

(e) Use the results from parts (c) and (d) to draw a rough sketch of the graph of this function.

Exercise 2.4

(a) Find the stationary points of the function

$$f(x) = 4x^3 + 6x^2 - 9x - 1.$$

(b) Classify each of the stationary points found in part (a), using the First Derivative Test.

(c) Find the y-coordinate of each of the stationary points on the graph of $y = f(x)$, and evaluate $f(0)$. Hence draw a rough sketch of the graph of this function.

Exercise 2.5

(a) Find the stationary points of the function

$$f(x) = x^4 - 8x^2 + 3.$$

(b) Classify each of the stationary points found in part (a), using the Second Derivative Test.

(c) Find the y-coordinate of each of the stationary points on the graph of $y = f(x)$. Hence draw a rough sketch of the graph of this function.

Exercise 2.6

Find the greatest and least values of the function

$$f(x) = x^3 - \tfrac{15}{2}x^2 + 12x + 2$$

(a) on the interval $[0, 6]$;

(b) on the interval $[-1, 3]$.

Section 3

Exercise 3.1

(a) Calculate the values of the ratio $(e^{-h} - 1)/h$ for each of $h = 0.1$, 0.001, $0.000\,01$, giving your answers to 6 decimal places. To what limit does the ratio seem to tend as $h \to 0$?

(b) Use your answer to part (a) to find the derivative of $f(x) = e^{-x}$ from definition (1.3) of Chapter C1 Section 1.

Exercise 3.2

Differentiate each of the following functions.

(a) $p(x) = 2x^5 - 7\sqrt{x}$

(b) $q(x) = 3\cos x - 2\sin x + 5$

(c) $r(x) = 4e^x - 5\ln x + 3x - \dfrac{2}{x^2}$

Exercise 3.3

Find each of the following derivatives, using the notation indicated.

(a) $\dfrac{dz}{dy}$ and $\dfrac{d^2z}{dy^2}$, where $z = 2e^y - 3\cos y$

(b) $\left.\dfrac{dw}{du}\right|_{u=2}$, where $w = \dfrac{3}{u} - 2\ln u + 4$

(c) $g'(t)$, where $g(t) = \cos t - 5\sin t$

(d) $\dot{s}$ and $\ddot{s}$, where $s = \frac{3}{2}t^2 - 7t + 4$

(e) $\dfrac{d}{dr}(3\pi r^2)$

Section 4

Exercise 4.1

Use the Product Rule to differentiate each of the following functions.

(a) $p(x) = 3x^2e^x$

(b) $f(x) = \left(x + \dfrac{1}{x}\right)\cos x$

(c) $g(t) = e^t(\sin t + \cos t)$

(d) $h(u) = (3 - 2u + u^2)\ln u$

Exercise 4.2

Use the Leibniz version of the Product Rule to find the following derivatives.

(a) $\dfrac{dy}{dx}$, where $y = x(\ln x - 1)$

(b) $\dfrac{dr}{dt}$, where $r = 2\sqrt{t}\cos t$

Exercise 4.3

Use the Quotient Rule to differentiate each of the following functions.

(a) $p(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(b) $f(t) = \dfrac{\cos t}{\sin t}$

(c) $z(y) = \dfrac{2\sin y}{y^2 - 4y + 3}$

Exercise 4.4

Use the Leibniz version of the Quotient Rule to find each of the following derivatives.

(a) $\dfrac{dy}{dx}$, where $y = \dfrac{\ln x}{e^x}$

(b) $\dfrac{ds}{dt}$, where $s = \dfrac{3e^t + 1}{t^2 - 1}$

Exercise 4.5

Use the Composite Rule to differentiate each of the following functions.

(a) $p(x) = \ln(\sin x)$

(b) $h(t) = \cos(t^4)$

(c) $f(x) = \cos^4 x$

(d) $k(t) = (t^2 - 3t + 1)^{15}$

Exercise 4.6

Use the Chain Rule to find each of the following derivatives.

(a) $\dfrac{dy}{dx}$, where $y = \sqrt{3x^2 + 2x + 1}$

(b) $\dfrac{dw}{dr}$, where $w = e^{-r^2}$

Exercise 4.7

Use Table 4.1 of Chapter C1 Section 4, together with the Product Rule, Quotient Rule or Composite Rule, to find each of the following derivatives.

(a) $f'(x)$, where $f(x) = e^{3x}\sin(2x)$

(b) $g'(t)$, where $g(t) = \dfrac{3t^2}{\ln(4t)}$

(c) $h'(x)$, where $h(x) = \cos^3(4x)$

(d) $\dfrac{dy}{dx}$, where $y = (x^3 + 2x - 1)\cos(5x)$

(e) $\dfrac{dy}{dx}$, where $y = \dfrac{e^{-2x}}{x^2}$

(f) $\dfrac{dz}{dt}$, where $z = \sin(e^{3t})$

Exercises for Chapter C2

Section 1

Exercise 1.1

Find each of the following indefinite integrals, using the results from Table 1.1 and the rules for integration stated in equations (1.2)–(1.4) of Chapter C2 Section 1, as appropriate.

(a) $\displaystyle\int \left(2x^3 - 3 + \frac{5}{x}\right) dx \quad (x > 0)$

(b) $\displaystyle\int 5\sin\left(\tfrac{1}{2}u\right) du$

(c) $\displaystyle\int \left(\tfrac{1}{3}e^{-2s} - \tfrac{1}{5}e^{-3s}\right) ds$

(d) $\displaystyle\int (\cos(2\pi t) + \sin(2\pi t))\, dt$

(e) $\displaystyle\int (3x^{5/2} - x^{3/2} + 5)\, dx \quad (x > 0)$

(f) $\displaystyle\int \left(\frac{2}{t} + \frac{3}{t^2} + \frac{4}{t^3}\right) dt \quad (t > 0)$

(g) $\displaystyle\int (7e^{y/3} - 2e^{-y/3})\, dy$

Section 2

Exercise 2.1

Find each of the following indefinite integrals. In each case, you will need to use an algebraic rearrangement before integrating.

(a) $\displaystyle\int \frac{t^5 - 1}{t^2\sqrt{t}}\, dt \quad (t > 0)$

(b) $\displaystyle\int e^{(x-1)/2}\, dx$

(c) $\displaystyle\int (x^2 - 1)(x^2 + 1)\, dx$

(d) $\displaystyle\int \frac{(u-3)(u-2)}{u}\, du \quad (u > 0)$

(e) $\displaystyle\int (e^{2x} - e^{-x})(e^x + 5)\, dx$

(f) $\displaystyle\int \left(x\sqrt{x} - \frac{1}{x\sqrt{x}}\right)\left(\sqrt{x} + \frac{2}{\sqrt{x}}\right) dx \quad (x > 0)$

Exercise 2.2

(a) Using a double-angle formula, find the indefinite integral

$$\int \sin x \cos x\, dx.$$

(b) Using an algebraic rearrangement, and the result of part (a), find the indefinite integral

$$\int (\sin x + 1)(\cos x - 2)\, dx.$$

Exercise 2.3

Using either of equations (2.3) and (2.4) of Chapter C2 Section 2, find each of the following indefinite integrals.

(a) $\displaystyle\int x^4(x^5 + 3)^3\, dx$

(b) $\displaystyle\int \frac{7t}{t^2 + 4}\, dt$

(c) $\displaystyle\int \sin x \cos x\, dx$

(d) $\displaystyle\int (e^u - e^{-u})^5(e^u + e^{-u})\, du$

(e) $\displaystyle\int \frac{3t + 1}{3t^2 + 2t + 1}\, dt$

(f) $\displaystyle\int \sin^4 x \cos x\, dx$

(g) $\displaystyle\int \frac{e^{-2t}}{3e^{-2t} + 2}\, dt$

(h) $\displaystyle\int \sqrt{x^3 + 3x + 2}\,(x^2 + 1)\, dx$

Exercise 2.4

(a) Using the Quotient Rule, find dy/dx when $y = \cot x$.

(b) Find the indefinite integral

$$\int \operatorname{cosec}^2 x \, dx.$$

(c) Using the trigonometric formula $\operatorname{cosec}^2 x = 1 + \cot^2 x$, and the result of part (b), find the indefinite integral

$$\int \cot^2 x \, dx.$$

(d) Using the result of part (a), and equation (2.3) of Chapter C2 Section 2, find the indefinite integral

$$\int \cot^2 x \operatorname{cosec}^2 x \, dx.$$

Exercise 2.5

(a) Show that

$$\frac{x^3}{x^2 - 1} = x + \frac{x}{x^2 - 1}.$$

Hence find the indefinite integral

$$\int \frac{x^3}{x^2 - 1} dx \quad (x > 1).$$

(b) By first multiplying the top and bottom of the fraction in the integrand by e^{2t}, and then applying equation (2.4) of Chapter C2 Section 2, find the indefinite integral

$$\int \frac{1}{7 + e^{-2t}} \, dt.$$

Exercise 2.6

Find each of the following indefinite integrals. You may need to use any of the methods introduced in Chapter C2 Section 2.

(a) $\displaystyle\int \frac{e^{2x} + e^{-2x}}{3e^x} \, dx$

(b) $\displaystyle\int \cot\theta \, d\theta \quad (0 < \theta < \pi)$

(c) $\displaystyle\int \sqrt{2 - t} \, dt \quad (t < 2)$

(d) $\displaystyle\int \sec^2 x \tan^2 x \, dx$

(Note that the derivative of $\tan x$ is $\sec^2 x$; see Activity 2.4(a) of Chapter C2 Section 2.)

(e) $\displaystyle\int \sec^4 x \, dx$

(Use the formula $\sec^2 x = 1 + \tan^2 x$; see equation (2.2) of Chapter C2 Section 2.)

(f) $\displaystyle\int \frac{1}{t + 1/t} \, dt$

Section 3

Exercise 3.1

A car, moving initially along a level road at velocity $20\,\mathrm{m\,s^{-1}}$, starts to climb a 1-in-10 hill without alteration to the power developed by its engine. Its acceleration while climbing is $-\frac{1}{10}g \simeq -1\,\mathrm{m\,s^{-2}}$.

(a) How far up the slope would the car travel before coming to rest, assuming that the slope were long enough for this to occur?

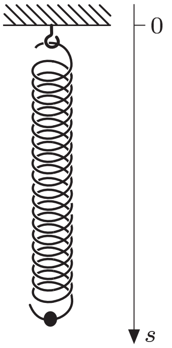
(b) In fact, the car reaches the summit of the hill after 150 metres. What is its velocity at the top of the hill?

(c) After reaching the summit, the car descends a 1-in-20 gradient down the other side. Again, the power output is unchanged, so that the car's acceleration is $\frac{1}{20}g \simeq 0.5\,\mathrm{m\,s^{-2}}$. At what distance down the slope does the velocity of the car reach $20\,\mathrm{m\,s^{-1}}$ again?

(d) What is the total time taken by the car to travel up the hill and down, until its velocity is $20\,\mathrm{m\,s^{-1}}$ again?

Exercise 3.2

A particle is attached to a spring, which hangs vertically from a hook on the ceiling. The particle is pulled downwards so that it is at distance 0.5 metres below the ceiling. At time $t = 0$ the particle is released, with initial velocity zero, and oscillates vertically. The s-axis is chosen to point vertically downwards, with the origin at the ceiling, as shown in the diagram below.



The acceleration of the particle after it starts to oscillate is given by the function

$$a = -2.5\cos(5t).$$

(a) Find an expression for the velocity v of the particle at time t.

(b) Find an expression for the position s of the particle at time t.

(c) What is the highest point reached by the particle, and at what time does this first occur?

(d) What is the maximum speed attained by the particle, and where does this occur?

Section 4

Exercise 4.1

Evaluate each of the following definite integrals. For parts (b), (c) and (d), give your answers to 3 decimal places.

(a) $\displaystyle\int_1^4 3\sqrt{x}\,dx$

(b) $\displaystyle\int_1^2 \left(x^2 - 3x + \frac{5}{x}\right) dx$

(c) $\displaystyle\int_{-1}^1 e^{2t}\,dt$

(d) $\displaystyle\int_0^2 \frac{x}{3x^2+1}\,dx$

(e) $\displaystyle\int_0^{\pi/2} \cos\left(\frac{x}{3}\right)\,dx$

Exercise 4.2

Which of the following definite integrals

$$\int_a^b f(x)\,dx$$

represents the area under the graph of $y = f(x)$ from $x = a$ to $x = b$?

(a) $\displaystyle\int_{-1}^1 (1+x^2)\,dx$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\,dx$

(c) $\displaystyle\int_0^5 (3-x)\,dx$

(d) $\displaystyle\int_{-1}^1 e^{-2x}\,dx$

Exercise 4.3

(a) Show that if $f(t) = t(t-2)(t-4)$ and $0 < t < 2$, then $f(t) > 0$. Hence find the area between the curve $y = f(t)$ and the t-axis from $t = 0$ to $t = 2$.

(b) Find the area under the graph of $\sin x \cos x$ from $x = 0$ to $x = \frac{1}{2}\pi$.

(Use the result of Exercise 2.2(a) or 2.3(c).)

(c) (i) Using the Product Rule, show that the derivative of $(x+1)e^{-x}$ is $-xe^{-x}$.

(ii) Hence find the area under the graph of xe^{-x} from $x = 0$ to $x = 2$, giving your answer to 3 decimal places.

Exercises for Chapter C3

Section 1

Exercise 1.1

In each case below, show that the given function satisfies the given differential equation.

(a) $y = \frac{1}{5}x^5 + 1;\quad \dfrac{dy}{dx} = x^4$

(b) $y = \sin^2 x;\quad \dfrac{dy}{dx} = 2\sin x\cos x$

(c) $y = 5e^{3x} + x;\quad \dfrac{dy}{dx} = 3y - 3x + 1$

(d) $y = e^{-x^2};\quad \dfrac{dy}{dx} = -2xy$

(e) $y = \tan x;\quad \dfrac{dy}{dx} = 1 + y^2$

(Note that the derivative of $\tan x$ is $\sec^2 x$. Also, use the formula $\sec^2 x = 1 + \tan^2 x$.)

(f) $y = x\ln x;\quad \dfrac{dy}{dx} = \dfrac{x+y}{x}$

Exercise 1.2

In each case below, find the general solution of the differential equation.

(a) $\dfrac{dy}{dx} = x^2 - 3x + 1$

(b) $\dfrac{dy}{dx} = \sin(2x) + \cos(2x)$

(c) $\dfrac{dv}{dt} = e^{-3t}$

(d) $\dfrac{dy}{dx} = x^3 - \dfrac{1}{x} + 3\sqrt{x} \quad (x > 0)$

Exercise 1.3

Solve each of the following initial-value problems.

(a) $\dfrac{dy}{dx} = 2 - 3\sin(4x),\quad y = \frac{1}{4}$ when $x = 0$

(b) $\dfrac{du}{dt} = \dfrac{1}{\sqrt{t}} - \dfrac{1}{t} \quad (t > 0),\quad u = 3$ when $t = 1$

(c) $\dfrac{dv}{dr} = 4\pi r^2,\quad v = 0$ when $r = 0$

(d) $\dfrac{ds}{dt} = \cos\left(\frac{1}{2}t\right),\quad s = 0$ when $t = \pi$

(e) $\dfrac{dy}{dt} = e^t + e^{2t},\quad y = 2$ when $t = 0$

Exercise 1.4

At time $t = 0$ a ball is thrown vertically upwards with speed $20\,\mathrm{m\,s^{-1}}$. In a model which incorporates the assumption that the ball is subject to air resistance proportional to its speed, the height s (in metres) of the ball above its point of projection at time t (in seconds) is given by the differential equation

$$\frac{ds}{dt} = 120e^{-0.1t} - 100.$$

(a) Find the general solution of this differential equation.

(b) Use the initial condition $s = 0$ when $t = 0$ to find the height of the ball at time t.

(c) Show that $ds/dt = 0$ when $t = 10\ln(\frac{6}{5})$, and hence find the maximum height reached by the ball, to the nearest centimetre.

Section 2

Exercise 2.1

(a) Show that the equation

$$\cos y = x$$

is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad (0 < y < \pi).$$

(You will need to use the equation $\cos^2 x + \sin^2 x = 1$.)

(b) Express the solution from part (a) in explicit form, and hence show that

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1).$$

Exercise 2.2

(a) Show that the equation

$$y + e^y = 2x - e^x + 2$$

is an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{2-e^x}{1+e^y}.$$

(b) What is the slope of the corresponding solution curve at the point $(0, 0)$?

Exercise 2.3

(a) Find, in implicit form, the general solution of the differential equation

$$\frac{1}{2y}\frac{dy}{dx} = -3x^2 \quad (y > 0).$$

(b) Find the corresponding explicit form of this general solution.

(c) Find the corresponding particular solution that satisfies the initial condition $y = 4$ when $x = 0$.

Exercise 2.4

Solve each of the following initial-value problems, giving your answers in explicit form.

(a) $\dfrac{dy}{dx} = 2xy^2 \quad (y > 0), \quad y = 1$ when $x = 0$

(b) $\dfrac{dx}{dt} = \dfrac{2(x-2)}{t} \quad (x > 2, t > 0),$
$x = 4$ when $t = 1$

(To integrate the function of x, use equation (2.4) of Chapter C2 Section 2.)

(c) $\dfrac{dy}{dx} = e^{-y}, \quad y = 1$ when $x = 0$

(d) $\dfrac{du}{dr} = 2r(1+u^2), \quad u = 1$ when $r = 0$

(To integrate the function of u, you may find that the Comment for Activity 2.2 of Chapter C3 Section 2 is useful.)

Exercise 2.5

To which of the following differential equations can one of the methods of direct integration or separation of variables be applied? For each equation for which one of these methods can be applied, use the appropriate method to find, in explicit form, the general solution of the differential equation.

(a) $\dfrac{dy}{dx} = 3\sec y \quad (-\frac{1}{2}\pi < y < \frac{1}{2}\pi)$

(b) $\dfrac{dy}{dx} = 3e^{2x}$

(c) $\dfrac{dy}{dx} = \cos^2 x + \cos^2 y$

(d) $\dfrac{du}{dt} = u + ut \quad (u > 0)$

(e) $\dfrac{dv}{dr} = e^{2rv}$

Exercise 2.6

In this exercise, we consider again a model in which a ball, thrown vertically upwards, is subject to air resistance proportional to its speed. The acceleration a of the ball is given by $a = -g - kv$, where v is its velocity, g is the magnitude of the acceleration due to gravity, and k is a constant. (Here v and a are taken to be positive in the upward direction.) This can be written as the differential equation

$$\frac{dv}{dt} = -k\left(v + \frac{g}{k}\right).$$

(a) Use the method of separation of variables to find the general solution of this equation.

(To integrate the function of v, use equation (2.4) of Chapter C2 Section 2, assuming that $v > -g/k$.)

(b) Find the particular solution satisfying the initial condition $v = v_0$ when $t = 0$.

(c) Taking $v_0 = 20$, $g = 10$ and $k = 0.1$, show that the solution from part (b) gives

$$v = 120e^{-0.1t} - 100$$

which is, with $v = ds/dt$, the equation given in Exercise 1.4 (for Chapter C3, in this booklet).

Section 3

Exercise 3.1

Cobalt-60 has a half-life of approximately 5.27 years.

(a) Find the value of the decay constant k for cobalt-60.

(b) Calculate the proportion of a sample of cobalt-60 which will still be present after 8 years.

(c) How long will it be before the proportion remaining is 10% of the original sample?

Exercise 3.2

The table below shows data for the radioactive decay of a sample of strontium-82 taken at 10-day intervals, where m_0 is the original mass of the sample and m is its mass after time t days.

t (days)	10	20	30	40	50	60
m/m_0	0.76	0.58	0.42	0.33	0.27	0.20
$\ln(m/m_0)$						

(a) Complete the third row of this table, giving entries to 2 decimal places.

(b) Plot a graph of $\ln(m/m_0)$ against t for these data. Draw 'by eye' a line through the origin which you think best fits the data points.

(c) Using your straight line from part (b), estimate values for the decay constant k and the half-life T for strontium-82.

(d) According to the exponential model, what proportion of the sample remains after 90 days?

Exercise 3.3

The population size of a country is currently 8 million, and its proportionate growth rate is 2% per year. Assume that the growth of this population is described by a (continuous) exponential model.

(a) Predict what the population size of the country will be after 10 years and after 40 years.

(b) After how long will the population size reach 80 million?

(c) What is the doubling time of the population?

Exercise 3.4

The table below shows data for the population of Bangladesh (in millions, to the nearest million) for the years 1950–2000.

Year	1950	1960	1970	1980	1990	2000
Population	42	52	66	85	110	137

(a) By using a log-linear plot for the data in this table, show that the exponential model is appropriate for the Bangladeshi population for the years 1950–2000. Measure time t in years, from $t = 0$ at the 1950 census date.

(b) Use your plot to estimate values for the proportionate growth rate K and the initial population size P_0 in the exponential model.

(c) Use this model to predict the population of Bangladesh in 2010.

Exercise 3.5

The cooling of a hot object surrounded by air may be modelled by assuming that the rate of decrease of the temperature θ (in °C) of the object at time t (in seconds) is proportional to $\theta - \theta_a$, where θ_a (in °C) is the constant temperature of the surrounding air. This model is described mathematically by the differential equation

$$\frac{d\theta}{dt} = -k(\theta - \theta_a), \qquad (1)$$

where k (in s^{-1}) is a positive constant. (This model is known as *Newton's Law of Cooling*.)

(a) By putting $y = \theta - \theta_a$, show that equation (1) is equivalent to the differential equation

$$\frac{dy}{dt} = -ky.$$

(b) Use result (3.9) of Chapter C3 Section 3 to write down the general solution of this differential equation for y. Hence, by putting $\theta = y + \theta_a$, find the general solution of equation (1).

(c) Find the corresponding particular solution of equation (1) for which $\theta = \theta_0$ when $t = 0$.

(d) Now suppose that the temperature of the surrounding air is 20°C, the initial temperature of the object is 90°C, and the object is found to have cooled to 60°C after 5 minutes.

Find the value of the constant k, and hence estimate the temperature of the object after a further 5 minutes.

Solutions for Chapter C1

Solution 1.1

We use result (1.1) of Chapter C1 Section 1, that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$.

(a) Here $f(x) = x^2 - 7x$, so $f'(x) = 2x - 7$.

Hence the gradient at the point $(9, 18)$, where $x = 9$, is $f'(9) = 18 - 7 = 11$.

(b) Here $f(x) = 2 - 3x^2$, so $f'(x) = -6x$.

Hence the gradient at the point $(2, -10)$, where $x = 2$, is $f'(2) = -12$.

(c) Now $f(x) = 7x^2 - 12x - 19$, so $f'(x) = 14x - 12$.

Hence the gradient at the point $(1, -24)$, where $x = 1$, is $f'(1) = 14 - 12 = 2$.

Solution 1.2

(a) We have $f(x) = 2x^2 - 5x + 1$, so $f'(x) = 4x - 5$.

(b) The gradient at the point $(2, -1)$, where $x = 2$, is $f'(2) = 3$.

(c) The line must pass through the point $(2, -1)$, and its slope must be the gradient of the graph at the point $(2, -1)$, that is, $f'(2) = 3$. Hence, from a result in Chapter A2 Section 1, its equation is $y - (-1) = 3(x - 2)$, which can be rearranged as $y = 3x - 7$.

(d) The gradient at the point (x, y) is $f'(x) = 4x - 5$. So the gradient is -9 when $4x - 5 = -9$; that is, when $x = -1$. The corresponding value of y is

$$\begin{aligned} y = f(-1) &= 2(-1)^2 - 5(-1) + 1 \\ &= 2 + 5 + 1 = 8, \end{aligned}$$

so the required point is $(-1, 8)$.

Solution 1.3

(a) Putting both fractions over the common denominator $x^2(x+h)^2$, we have

$$\begin{aligned} \frac{1}{(x+h)^2} - \frac{1}{x^2} &= \frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2} \\ &= \frac{x^2 - (x+h)^2}{x^2(x+h)^2} \\ &= \frac{x^2 - (x^2 + 2hx + h^2)}{x^2(x+h)^2} \\ &= \frac{-2hx - h^2}{x^2(x+h)^2} \\ &= -\frac{h(2x+h)}{x^2(x+h)^2}. \end{aligned}$$

(b) By equation (1.3), and using the result of part (a), we consider the quotient

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right) \\ &= \frac{1}{h}\left(-\frac{h(2x+h)}{x^2(x+h)^2}\right) \\ &= -\frac{(2x+h)}{x^2(x+h)^2}. \end{aligned}$$

For small values of h, this quotient is close to

$$-\frac{2x}{x^4} = -\frac{2}{x^3}.$$

Hence $f'(x) = -2/x^3$.

(c) Result (1.4) states that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Here $f(x) = 1/x^2 = x^{-2}$, so that $n = -2$. The derivative is therefore given by

$$f'(x) = (-2)x^{-2-1} = -2x^{-3},$$

which is often written as $f'(x) = -2/x^3$, in agreement with the result of part (b).

Solution 1.4

In each case we apply result (1.4), which states that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

(a) Here $f(x) = x^{17}$, so $n = 17$. The derivative is

$$f'(x) = 17x^{17-1} = 17x^{16}.$$

(b) Here $f(x) = 1/x^{17} = x^{-17}$, so $n = -17$. The derivative is

$$f'(x) = -17x^{-17-1} = -17x^{-18},$$

which is often written as $f'(x) = -17/x^{18}$.

(c) Here $f(x) = 1/\sqrt{x} = 1/x^{1/2} = x^{-1/2}$, so $n = -\frac{1}{2}$. The derivative is

$$f'(x) = -\tfrac{1}{2}x^{(-1/2)-1} = -\tfrac{1}{2}x^{-3/2},$$

which is often written as

$$f'(x) = -\frac{1}{2x^{3/2}} \text{ or } -\frac{1}{2x\sqrt{x}}.$$

(d) Here $f(x) = x\sqrt{x} = x^{3/2}$, so $n = \frac{3}{2}$. The derivative is

$$f'(x) = \tfrac{3}{2}x^{(3/2)-1} = \tfrac{3}{2}x^{1/2} = \tfrac{3}{2}\sqrt{x}.$$

Solution 1.5

(a) The position function is $s = f(t) = 15t - 5t^2$. This is a quadratic function of t, so we apply result (1.1) to obtain the velocity function

$$v = f'(t) = 15 - 10t.$$

(b) When $t = 1$, we have $v = 15 - 10 = 5$, so the velocity after 1 second is $5\,\mathrm{m\,s^{-1}}$.

(c) When $v = 0$, we have $0 = 15 - 10t$, so $t = \frac{3}{2}$. Thus the ball has velocity zero after 1.5 seconds.

(d) From part (c), $v = 0$ when $t = \frac{3}{2}$. At this time

$$s = 15\left(\tfrac{3}{2}\right) - 5\left(\tfrac{3}{2}\right)^2 = \tfrac{45}{2} - \tfrac{45}{4} = \tfrac{45}{4} = 11.25.$$

So the height of the ball when its velocity is zero is 11.25 metres.

Solution 2.1

For each part we use results (1.4) and (2.3) of Chapter C1.

(a) The derivative of the polynomial function $f(x) = \frac{3}{2}x^{10} - 2x^6 + 3x$ is

$$\begin{aligned} f'(x) &= \tfrac{3}{2}(10x^9) - 2(6x^5) + 3(1) \\ &= 15x^9 - 12x^5 + 3. \end{aligned}$$

(b) The derivative of the polynomial function $f(x) = 7 + 4x^2 - 3x^4 + x^6$ is

$$\begin{aligned} f'(x) &= 4(2x) - 3(4x^3) + 6x^5 \\ &= 8x - 12x^3 + 6x^5. \end{aligned}$$

Solution 2.2

(a) The derivative of the function $f(x) = \frac{1}{4}x^8 - \frac{1}{3}x^6 + \frac{1}{3}$ is

$$\begin{aligned} f'(x) &= \tfrac{1}{4}(8x^7) - \tfrac{1}{3}(6x^5) \\ &= 2x^7 - 2x^5. \end{aligned}$$

The point $(2, 43)$ on the graph of $y = f(x)$ corresponds to $x = 2$, for which the gradient is

$$\begin{aligned} f'(2) &= 2(2^7) - 2(2^5) \\ &= 256 - 64 = 192. \end{aligned}$$

(b) The derivative of the function $f(x) = x^5 - 3x^3 - x^2 - 6$ is

$$\begin{aligned} f'(x) &= 5x^4 - 3(3x^2) - (2x) \\ &= 5x^4 - 9x^2 - 2x. \end{aligned}$$

The point $(-1, -5)$ on the graph corresponds to $x = -1$, for which the gradient is

$$\begin{aligned} f'(-1) &= 5(-1)^4 - 9(-1)^2 - 2(-1) \\ &= 5 - 9 + 2 = -2. \end{aligned}$$

Solution 2.3

(a) The derivative of the function $f(x) = 3x^2 + 18x + 15$ is $f'(x) = 6x + 18$.

(b) We have $f'(-3) = 6(-3) + 18 = 0$.

For $x < -3$, we have $x + 3 < 0$, so $f'(x) = 6x + 18 = 6(x + 3)$ is the product of 6 and a negative number, and so $f'(x) < 0$.

For $x > -3$, we have $x + 3 > 0$, so $f'(x)$ is the product of 6 and a positive number, and so $f'(x) > 0$.

(c) By the Increasing/Decreasing Criterion, f is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$.

(d) We have $f(0) = 15$, and

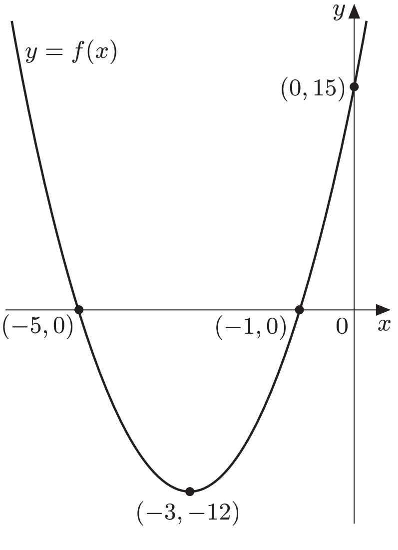
$$f(-3) = 3(-3)^2 + 18(-3) + 15 = -12.$$

Also $f(x) = 0$ when $3x^2 + 18x + 15 = 0$, or

$$3(x^2 + 6x + 5) = 3(x + 1)(x + 5) = 0,$$

so $f(x) = 0$ when $x = -1$ or $x = -5$.

(e) Using the results from parts (c) and (d), the graph passes through the points $(-5, 0)$, $(-3, -12)$, $(-1, 0)$ and $(0, 15)$. It is decreasing for $x < -3$ and increasing for $x > -3$, so takes its least value, -12, at $x = -3$. A sketch of the graph is as follows.



(You may like to compare this solution with that to Exercise 2.3(a) for Chapter A3 in Exercise Booklet A, where you sketched the same graph using a scaling and translations, starting from the graph of $y = x^2$.)

Solution 2.4

(a) The derivative of the function $f(x) = 4x^3 + 6x^2 - 9x - 1$ is

$$f'(x) = 12x^2 + 12x - 9 = 3(4x^2 + 4x - 3)$$
$$= 3(2x - 1)(2x + 3).$$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are at $x = \frac{1}{2}$ and $x = -\frac{3}{2}$.

(b) To classify the stationary point at $x = -\frac{3}{2}$ using the First Derivative Test, choose nearby points to the left and right of $-\frac{3}{2}$, say $x_{\rm L} = -2$ and $x_{\rm R} = 0$ (which is closer to $-\frac{3}{2}$ than the next stationary point, at $\frac{1}{2}$). Then we have

$$f'(x_{\rm L}) = 48 - 24 - 9 = 15 > 0,$$
$$f'(x_{\rm R}) = f'(0) = -9 < 0,$$

so f has a local maximum at $x = -\frac{3}{2}$.

To classify the stationary point at $x = \frac{1}{2}$ using the First Derivative Test, choose nearby points to the left and right of $\frac{1}{2}$, say $x_{\rm L} = 0$ and $x_{\rm R} = 1$. Then we have

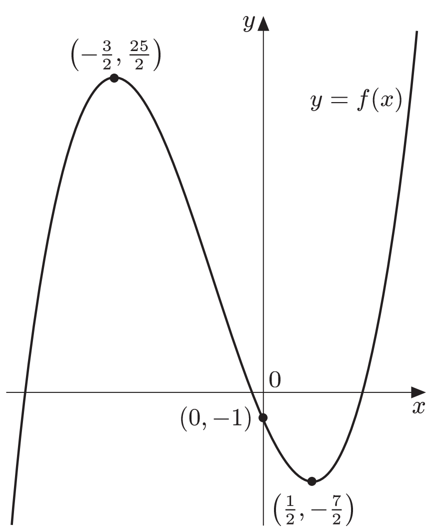
$$f'(x_{\rm L}) = f'(0) = -9 < 0,$$
$$f'(x_{\rm R}) = f'(1) = 12 + 12 - 9 = 15 > 0,$$

so f has a local minimum at $x = \frac{1}{2}$.

(c) The y-coordinates of stationary points on the graph of $y = f(x)$ are

$$f(-\tfrac{3}{2}) = 4(-\tfrac{27}{8}) + 6(\tfrac{9}{4}) - 9(-\tfrac{3}{2}) - 1 = \tfrac{25}{2},$$
$$f(\tfrac{1}{2}) = 4(\tfrac{1}{8}) + 6(\tfrac{1}{4}) - 9(\tfrac{1}{2}) - 1 = -\tfrac{7}{2}.$$

Also $f(0) = -1$. Hence the graph passes through the points $(-\frac{3}{2}, \frac{25}{2})$, a local maximum, and $(\frac{1}{2}, -\frac{7}{2})$, a local minimum. It also passes through $(0, -1)$, that is, it cuts the y-axis at $y = -1$. A sketch of the graph is as follows.



Solution 2.5

(a) The derivative of the function $f(x) = x^4 - 8x^2 + 3$ is

$$f'(x) = 4x^3 - 16x$$
$$= 4x(x^2 - 4)$$
$$= 4x(x + 2)(x - 2).$$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are at $x = -2$, $x = 0$ and $x = 2$.

(b) The second derivative of f is

$$f''(x) = 12x^2 - 16.$$

For the stationary point at $x = -2$, we have

$$f''(-2) = 12(-2)^2 - 16 = 32 > 0,$$

so f has a local minimum at $x = -2$, by the Second Derivative Test.

For the stationary point at $x = 0$, we have

$$f''(0) = -16 < 0,$$

so f has a local maximum at $x = 0$, by the Second Derivative Test.

For the stationary point at $x = 2$, we have

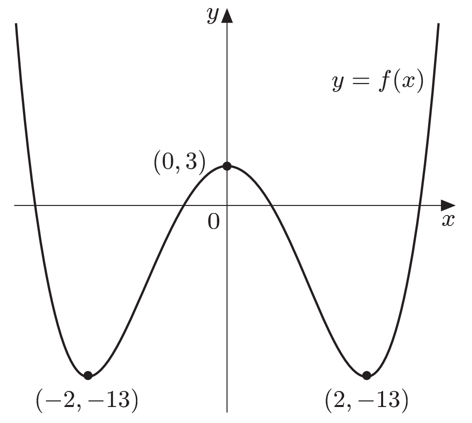
$$f''(2) = 12(2)^2 - 16 = 32 > 0,$$

so f has a local minimum at $x = 2$, by the Second Derivative Test.

(c) The y-coordinates of the stationary points on the graph of $y = f(x)$ are, respectively,

$$f(-2) = (-2)^4 - 8(-2)^2 + 3 = -13,$$
$$f(0) = 3, \quad f(2) = 2^4 - 8(2)^2 + 3 = -13.$$

Hence the graph passes through the points $(-2, -13)$, a local minimum, $(0, 3)$, a local maximum, and $(2, -13)$, a local minimum. A sketch of the graph is as follows.



Solution 2.6

(a) We follow the steps of the Optimisation Procedure.

Step 1: The derivative of the function $f(x) = x^3 - \frac{15}{2}x^2 + 12x + 2$ is

$$\begin{aligned} f'(x) &= 3x^2 - 15x + 12 \\ &= 3(x^2 - 5x + 4) \\ &= 3(x-1)(x-4). \end{aligned}$$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are at $x = 1$ and $x = 4$.

Step 2: At the endpoints of the interval $I = [0, 6]$, the function values are

$$\begin{aligned} &f(0) = 2 \quad \text{and} \\ &f(6) = 216 - 270 + 72 + 2 = 20, \end{aligned}$$

while at the stationary points (which are both inside I) the function values are

$$\begin{aligned} &f(1) = 1 - \tfrac{15}{2} + 12 + 2 = \tfrac{15}{2}, \\ &f(4) = 64 - 120 + 48 + 2 = -6. \end{aligned}$$

Step 3: The greatest value of $f(x)$ on $[0, 6]$ is 20 (at $x = 6$) and the least value is -6 (at $x = 4$).

(b) We again follow the steps of the Optimisation Procedure, but Step 1 has already been completed in part (a).

Step 2: At the endpoints of the interval $I = [-1, 3]$, the function values are

$$\begin{aligned} &f(-1) = -1 - \tfrac{15}{2} - 12 + 2 = -\tfrac{37}{2}, \\ &f(3) = 27 - \tfrac{135}{2} + 36 + 2 = -\tfrac{5}{2}. \end{aligned}$$

The only stationary point inside I is at $x = 1$, and $f(1) = \frac{15}{2}$ as in part (a).

Step 3: The greatest value of $f(x)$ on $[-1, 3]$ is $\frac{15}{2}$ (at $x = 1$) and the least value is $-\frac{37}{2}$ (at $x = -1$).

Solution 3.1

(a) Evaluating the ratio $(e^{-h} - 1)/h$ for each of $h = 0.1$, 0.001, $0.000\,01$ in turn gives the respective answers (to 6 d.p.) $-0.951\,626$, $-0.999\,500$, $-0.999\,995$. The limiting value as $h \to 0$ appears to be -1.

(b) If $f(x) = e^{-x}$, then

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{e^{-(x+h)} - e^{-x}}{h} \\ &= \frac{e^{-x}e^{-h} - e^{-x}}{h} \quad \text{(by a law of indices)} \\ &= e^{-x}\left(\frac{e^{-h} - 1}{h}\right). \end{aligned}$$

On taking the limit as $h \to 0$, using definition (1.3),

$$\begin{aligned} f'(x) &= \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right) \\ &= e^{-x}\lim_{h \to 0}\left(\frac{e^{-h} - 1}{h}\right) \\ &= -e^{-x} \text{ (using the result of part (a)).} \end{aligned}$$

Solution 3.2

In each case, we apply the Sum and Constant Multiple Rules in combination.

(a) Since $p(x) = 2x^5 - 7\sqrt{x} = 2x^5 - 7x^{1/2}$, the derivative is

$$\begin{aligned} p'(x) &= 10x^4 - \tfrac{7}{2}x^{-1/2} \\ &= 10x^4 - \frac{7}{2\sqrt{x}}. \end{aligned}$$

(b) The derivative of $q(x) = 3\cos x - 2\sin x + 5$ is

$$q'(x) = -3\sin x - 2\cos x.$$

(c) The derivative of $r(x) = 4e^x - 5\ln x + 3x - 2/x^2$ is (since $-2/x^2 = -2x^{-2}$)

$$r'(x) = 4e^x - \frac{5}{x} + 3 + \frac{4}{x^3}.$$

Solution 3.3

(a) The first and second derivatives of $z = 2e^y - 3\cos y$ with respect to y are

$$\frac{dz}{dy} = 2e^y + 3\sin y, \qquad \frac{d^2z}{dy^2} = 2e^y + 3\cos y.$$

(b) The derivative of $w = 3/u - 2\ln u + 4$ with respect to u is

$$\frac{dw}{du} = -\frac{3}{u^2} - \frac{2}{u},$$

so that

$$\left.\frac{dw}{du}\right|_{u=2} = -\tfrac{3}{4} - 1 = -\tfrac{7}{4}.$$

(c) The derivative of $g(t) = \cos t - 5\sin t$ is

$$g'(t) = -\sin t - 5\cos t.$$

(d) The first and second derivatives of $s = \frac{3}{2}t^2 - 7t + 4$ with respect to t are

$$\dot{s} = 3t - 7, \quad \ddot{s} = 3.$$

(e) The derivative of $3\pi r^2$ with respect to r is

$$\frac{d}{dr}(3\pi r^2) = 6\pi r.$$

Solution 4.1

(a) The derivative of $p(x) = 3x^2e^x$ is

$$\begin{aligned} p'(x) &= (6x)(e^x) + (3x^2)(e^x) \\ &= 3x(2+x)e^x. \end{aligned}$$

(b) The derivative of $f(x) = (x + 1/x)\cos x$, that is, of $f(x) = (x + x^{-1})\cos x$, is

$$\begin{aligned} f'(x) &= (1 - x^{-2})(\cos x) + (x + x^{-1})(-\sin x) \\ &= \left(1 - \frac{1}{x^2}\right)\cos x - \left(x + \frac{1}{x}\right)\sin x. \end{aligned}$$

(c) The derivative of $g(t) = e^t(\sin t + \cos t)$ is

$$\begin{aligned} g'(t) &= (e^t)(\sin t + \cos t) + (e^t)(\cos t + (-\sin t)) \\ &= e^t(\sin t + \cos t + \cos t - \sin t) \\ &= 2e^t \cos t. \end{aligned}$$

(d) The derivative of $h(u) = (3 - 2u + u^2)\ln u$ is

$$\begin{aligned} h'(u) &= (-2 + 2u)(\ln u) + (3 - 2u + u^2)\frac{1}{u} \\ &= (2u - 2)\ln u + \frac{3}{u} - 2 + u. \end{aligned}$$

Solution 4.2

(a) We have $y = uv$, where $u = x$ and $v = \ln x - 1$, so

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= (1)(\ln x - 1) + x\left(\frac{1}{x}\right) \\ &= \ln x - 1 + 1 \\ &= \ln x. \end{aligned}$$

(b) We have $r = uv$, where $u = 2\sqrt{t} = 2t^{1/2}$ and $v = \cos t$, so

$$\begin{aligned} \frac{dr}{dt} &= \frac{du}{dt}v + u\frac{dv}{dt} \\ &= \left(2 \times \tfrac{1}{2}t^{-1/2}\right)(\cos t) + (2t^{1/2})(-\sin t) \\ &= \frac{\cos t}{\sqrt{t}} - 2\sqrt{t}\sin t. \end{aligned}$$

Solution 4.3

(a) The derivative of $p(x) = (x^2 - 1)/(x^2 + 1)$ is

$$\begin{aligned} p'(x) &= \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} \\ &= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} \\ &= \frac{4x}{(x^2 + 1)^2}. \end{aligned}$$

(b) The derivative of $f(t) = \cos t/\sin t$ is

$$\begin{aligned} f'(t) &= \frac{(\sin t)(-\sin t) - (\cos t)(\cos t)}{(\sin t)^2} \\ &= \frac{-\sin^2 t - \cos^2 t}{\sin^2 t} \\ &= -\frac{1}{\sin^2 t} \\ &\qquad \text{(using the result } \cos^2 t + \sin^2 t = 1). \end{aligned}$$

(Since $\cos t/\sin t = \cot t$ and $1/\sin t = \operatorname{cosec} t$, this shows that the derivative of $\cot t$ is $-\operatorname{cosec}^2 t$.)

(c) The derivative of $z(y) = 2\sin y/(y^2 - 4y + 3)$ is

$$\begin{aligned} z'(y) &= \frac{(y^2 - 4y + 3)(2\cos y) - (2\sin y)(2y - 4)}{(y^2 - 4y + 3)^2} \\ &= \frac{2(y^2 - 4y + 3)\cos y - 4(y - 2)\sin y}{(y^2 - 4y + 3)^2}. \end{aligned}$$

Solution 4.4

(a) We have $y = u/v$, where $u = \ln x$ and $v = e^x$, so

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \\ &= \frac{1}{(e^x)^2}\left((e^x)\left(\frac{1}{x}\right) - (\ln x)(e^x)\right) \\ &= \frac{e^x}{(e^x)^2}\left(\frac{1}{x} - \ln x\right) \\ &= \frac{1}{e^x}\left(\frac{1}{x} - \ln x\right). \end{aligned}$$

(b) We have $s = u/v$, where $u = 3e^t + 1$ and $v = t^2 - 1$, so

$$\begin{aligned} \frac{ds}{dt} &= \frac{1}{v^2}\left(v\frac{du}{dt} - u\frac{dv}{dt}\right) \\ &= \frac{1}{(t^2 - 1)^2}\left((t^2 - 1)(3e^t) - (3e^t + 1)(2t)\right) \\ &= \frac{3e^t(t^2 - 2t - 1) - 2t}{(t^2 - 1)^2}. \end{aligned}$$

Solution 4.5

(a) The derivative of $p(x) = \ln(\sin x)$ is

$$p'(x) = \left(\frac{1}{\sin x}\right)(\cos x) = \frac{\cos x}{\sin x} = \cot x.$$

Expanded version: Here $p(x) = \ln(\sin x) = g(f(x))$, where $u = f(x) = \sin x$ and $g(u) = \ln u$.

Now $f'(x) = \cos x$ and $g'(u) = 1/u$, so we have

$$\begin{aligned} p'(x) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \quad (\text{where } u = f(x) = \sin x) \\ &= \left(\frac{1}{u}\right)(\cos x) \\ &= \frac{1}{\sin x}\cos x \quad (\text{since } u = \sin x) \\ &= \cot x. \end{aligned}$$

(b) The derivative of $h(t) = \cos(t^4)$ is

$$\begin{aligned} h'(t) &= (-\sin(t^4))(4t^3) \\ &= -4t^3\sin(t^4). \end{aligned}$$

Expanded version: Here $h(t) = \cos(t^4) = g(f(t))$, where $u = f(t) = t^4$ and $g(u) = \cos u$. Now $f'(t) = 4t^3$ and $g'(u) = -\sin u$, so we have

$$\begin{aligned} h'(t) &= g'(f(t))f'(t) \\ &= g'(u)f'(t) \quad (\text{where } u = f(t) = t^4) \\ &= (-\sin u)(4t^3) \\ &= -\sin(t^4)(4t^3) \quad (\text{since } u = t^4) \\ &= -4t^3\sin(t^4). \end{aligned}$$

(c) The derivative of $f(x) = \cos^4 x = (\cos x)^4$ is

$$\begin{aligned} f'(x) &= 4(\cos x)^3(-\sin x) \\ &= -4\cos^3 x \sin x. \end{aligned}$$

Expanded version: Here $f(x) = \cos^4 x = (\cos x)^4 = q(p(x))$, where $u = p(x) = \cos x$ and $q(u) = u^4$. Now $p'(x) = -\sin x$ and $q'(u) = 4u^3$, so we have

$$\begin{aligned} f'(x) &= q'(p(x))p'(x) \\ &= q'(u)p'(x) \quad (\text{where } u = p(x) = \cos x) \\ &= (4u^3)(-\sin x) \\ &= -4\cos^3 x \sin x \quad (\text{since } u = \cos x). \end{aligned}$$

(d) The derivative of $k(t) = (t^2 - 3t + 1)^{15}$ is

$$k'(t) = 15(t^2 - 3t + 1)^{14}(2t - 3).$$

Expanded version: Here $k(t) = g(f(t))$, where $u = f(t) = t^2 - 3t + 1$ and $g(u) = u^{15}$. Now $f'(t) = 2t - 3$ and $g'(u) = 15u^{14}$, so we have

$$\begin{aligned} k'(t) &= g'(f(t))f'(t) \\ &= g'(u)f'(t) \quad (\text{where } u = f(t) = t^2 - 3t + 1) \\ &= (15u^{14})(2t - 3) \\ &= 15(t^2 - 3t + 1)^{14}(2t - 3) \\ &\qquad (\text{since } u = t^2 - 3t + 1). \end{aligned}$$

Solution 4.6

(a) Here $y = \sqrt{3x^2 + 2x + 1} = \sqrt{u} = u^{1/2}$, where $u = 3x^2 + 2x + 1$, so we have

$$\frac{dy}{du} = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}, \quad \frac{du}{dx} = 6x + 2$$

and

$$\begin{aligned} \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} &= \left(\frac{1}{2\sqrt{u}}\right)(6x + 2) \\ &= \frac{1}{2\sqrt{3x^2 + 2x + 1}}(6x + 2) \\ &= \frac{3x + 1}{\sqrt{3x^2 + 2x + 1}}. \end{aligned}$$

(b) Here $w = e^{-r^2} = e^u$, where $u = -r^2$. Hence

$$\frac{dw}{du} = e^u \quad \text{and} \quad \frac{du}{dr} = -2r,$$

so that

$$\frac{dw}{dr} = \frac{dw}{du}\frac{du}{dr} = (e^u)(-2r) = -2re^{-r^2}.$$

Solution 4.7

(a) Using the Product Rule, where $f(x) = e^{3x}\sin(2x)$, we have

$$\begin{aligned} f'(x) &= (3e^{3x})(\sin(2x)) + (e^{3x})(2\cos(2x)) \\ &= e^{3x}(3\sin(2x) + 2\cos(2x)). \end{aligned}$$

(b) Using the Quotient Rule, where $g(t) = 3t^2/\ln(4t)$, we have

$$\begin{aligned} g'(t) &= \frac{(\ln(4t))(6t) - (3t^2)(1/t)}{(\ln(4t))^2} \\ &= \frac{6t\ln(4t) - 3t}{(\ln(4t))^2}. \end{aligned}$$

(c) Using the Composite Rule, where $h(x) = \cos^3(4x) = (\cos(4x))^3$, we have

$$\begin{aligned} h'(x) &= 3(\cos(4x))^2(-4\sin(4x)) \\ &= -12\cos^2(4x)\sin(4x). \end{aligned}$$

(d) Using the Leibniz form of the Product Rule, where $y = (x^3 + 2x - 1)\cos(5x)$, we have

$$\begin{aligned} \frac{dy}{dx} &= (3x^2 + 2)\cos(5x) + (x^3 + 2x - 1)(-5\sin(5x)) \\ &= (3x^2 + 2)\cos(5x) - 5(x^3 + 2x - 1)\sin(5x). \end{aligned}$$

(e) Using the Leibniz form of the Quotient Rule, where $y = e^{-2x}/x^2$, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{(x^2)^2}\left(x^2(-2e^{-2x}) - e^{-2x}(2x)\right) \\ &= \frac{e^{-2x}(-2x^2 - 2x)}{x^4} \\ &= -\frac{2e^{-2x}(x + 1)}{x^3}. \end{aligned}$$

(f) Using the Chain Rule (the Leibniz form of the Composite Rule), where $z = \sin(e^{3t})$, we have

$$\begin{aligned} \frac{dz}{dt} &= \frac{dz}{du}\frac{du}{dt} \quad (\text{where } z = \sin u \text{ and } u = e^{3t}) \\ &= (\cos u)(3e^{3t}) \\ &= \cos(e^{3t})(3e^{3t}) = 3e^{3t}\cos(e^{3t}). \end{aligned}$$

Solutions for Chapter C2

Solution 1.1

In each solution, c is an arbitrary constant.

(a) $\displaystyle\int \left(2x^3 - 3 + \frac{5}{x}\right) dx = \tfrac{1}{2}x^4 - 3x + 5\ln x + c$

(b) $\displaystyle\int 5\sin\left(\tfrac{1}{2}u\right) du = -10\cos\left(\tfrac{1}{2}u\right) + c$

(c) $\displaystyle\int \left(\tfrac{1}{3}e^{-2s} - \tfrac{1}{5}e^{-3s}\right) ds = -\tfrac{1}{6}e^{-2s} + \tfrac{1}{15}e^{-3s} + c$

(d) $$\int (\cos(2\pi t) + \sin(2\pi t))\, dt = \frac{1}{2\pi}\sin(2\pi t) - \frac{1}{2\pi}\cos(2\pi t) + c$$

(e) $\displaystyle\int (3x^{5/2} - x^{3/2} + 5)\, dx = \tfrac{6}{7}x^{7/2} - \tfrac{2}{5}x^{5/2} + 5x + c$

(f) $$\begin{aligned}\int \left(\frac{2}{t} + \frac{3}{t^2} + \frac{4}{t^3}\right) dt &= \int \left(\frac{2}{t} + 3t^{-2} + 4t^{-3}\right) dt \\ &= 2\ln t - 3t^{-1} - 2t^{-2} + c \\ &= 2\ln t - \frac{3}{t} - \frac{2}{t^2} + c\end{aligned}$$

(g) $\displaystyle\int (7e^{y/3} - 2e^{-y/3})\, dy = 21e^{y/3} + 6e^{-y/3} + c$

Solution 2.1

In each solution, c is an arbitrary constant.

(a) $$\begin{aligned}\int \frac{t^5 - 1}{t^2\sqrt{t}}\, dt &= \int \frac{t^5 - 1}{t^{5/2}}\, dt \\ &= \int \left(t^{5/2} - t^{-5/2}\right) dt \\ &= \tfrac{2}{7}t^{7/2} + \tfrac{2}{3}t^{-3/2} + c\end{aligned}$$

(b) $$\begin{aligned}\int e^{(x-1)/2}\, dx &= \int e^{x/2}e^{-1/2}\, dx \\ &= e^{-1/2} \times 2e^{x/2} + c \\ &= 2e^{(x-1)/2} + c\end{aligned}$$

(c) $$\begin{aligned}\int (x^2 - 1)(x^2 + 1)\, dx &= \int (x^4 - 1)\, dx \\ &= \tfrac{1}{5}x^5 - x + c\end{aligned}$$

(d) $$\begin{aligned}\int \frac{(u-3)(u-2)}{u}\, du &= \int \frac{u^2 - 5u + 6}{u}\, du \\ &= \int \left(u - 5 + \frac{6}{u}\right) du \\ &= \tfrac{1}{2}u^2 - 5u + 6\ln u + c\end{aligned}$$

(e) $$\begin{aligned}&\int (e^{2x} - e^{-x})(e^x + 5)\, dx \\ &= \int (e^{3x} + 5e^{2x} - 1 - 5e^{-x})\, dx \\ &= \tfrac{1}{3}e^{3x} + \tfrac{5}{2}e^{2x} - x + 5e^{-x} + c\end{aligned}$$

(f) $$\begin{aligned}&\int \left(x\sqrt{x} - \frac{1}{x\sqrt{x}}\right)\left(\sqrt{x} + \frac{2}{\sqrt{x}}\right) dx \\ &= \int \left(x^2 + 2x - \frac{1}{x} - \frac{2}{x^2}\right) dx \\ &= \int \left(x^2 + 2x - \frac{1}{x} - 2x^{-2}\right) dx \\ &= \tfrac{1}{3}x^3 + x^2 - \ln x + 2x^{-1} + c\end{aligned}$$

Solution 2.2

(a) Using the formula $\sin(2\theta) = 2\sin\theta\cos\theta$ with $\theta = x$, we have

$$\sin x\cos x = \tfrac{1}{2}\sin(2x),$$

so that

$$\begin{aligned}\int \sin x\cos x\, dx &= \int \tfrac{1}{2}\sin(2x) dx \\ &= -\tfrac{1}{4}\cos(2x) + c,\end{aligned}$$

where c is an arbitrary constant.

(b) We have

$$\begin{aligned}&\int (\sin x + 1)(\cos x - 2) dx \\ &= \int (\sin x\cos x - 2\sin x + \cos x - 2) dx \\ &= -\tfrac{1}{4}\cos(2x) + 2\cos x + \sin x - 2x + c,\end{aligned}$$

where c is an arbitrary constant.

Solution 2.3

(a) In the integrand $x^4(x^5+3)^3$, the factor x^4 is, except for a constant multiple, the derivative of $x^5 + 3$. Hence we can apply equation (2.3) with $f(x) = x^5 + 3$ and $n = 3$. Since $f'(x) = 5x^4$, we write $x^4 = \frac{1}{5}(5x^4)$ before applying the formula. Thus we have

$$\begin{aligned}\int x^4(x^5+3)^3 dx &= \tfrac{1}{5}\int (x^5+3)^3(5x^4)\, dx \\ &= \tfrac{1}{5} \times \tfrac{1}{4}(x^5+3)^4 + c \\ &= \tfrac{1}{20}(x^5+3)^4 + c,\end{aligned}$$

where c is an arbitrary constant.

(b) The integrand is $7t/(t^2+4)$. Here the numerator $7t$ is, except for a constant multiple, the derivative of the denominator $t^2 + 4$. Also $t^2 + 4 > 0$ for all t, so we can apply equation (2.4) with $f(t) = t^2 + 4$. Since $f'(t) = 2t$, we write $7t = \frac{7}{2}(2t)$ before applying the formula. Thus we have

$$\int \frac{7t}{t^2+4}\, dt = \tfrac{7}{2}\int \frac{2t}{t^2+4}\, dt = \tfrac{7}{2}\ln(t^2+4) + c,$$

where c is an arbitrary constant.

(c) The factor $\cos x$ in the integrand is the derivative of $\sin x$, so we can apply equation (2.3) with $f(x) = \sin x$ and $n = 1$. Thus we have

$$\int \sin x \cos x \, dx = \tfrac{1}{2} \sin^2 x + c,$$

where c is an arbitrary constant.

Alternatively, since $\sin x$ is, except for a constant multiple, the derivative of $\cos x$, we can apply equation (2.3) with $f(x) = \cos x$ and $n = 1$. Since $f'(x) = -\sin x$, we write $\sin x$ as $-(-\sin x)$ before applying the formula. Thus we have

$$\begin{aligned}\int \sin x \cos x \, dx &= -\int \cos x(-\sin x)\, dx \\ &= -\tfrac{1}{2}\cos^2 x + c,\end{aligned}$$

where c is an arbitrary constant.

(You might like to show, by using trigonometric formulas given in Chapter C2 Section 2, that the two apparently different answers obtained here, and the answer found in Exercise 2.2(a), are all correct expressions for this indefinite integral.)

(d) The factor $e^u + e^{-u}$ in the integrand is the derivative of $e^u - e^{-u}$, so we can apply equation (2.3) with $f(u) = e^u - e^{-u}$ and $n = 5$. Thus we have

$$\int (e^u - e^{-u})^5 (e^u + e^{-u})\, du = \tfrac{1}{6}(e^u - e^{-u})^6 + c,$$

where c is an arbitrary constant.

(e) The integrand is $(3t+1)/(3t^2+2t+1)$. Here the numerator $3t+1$ is, except for a constant multiple, the derivative of the denominator $3t^2+2t+1$. Also $3t^2+2t+1 > 0$ for all t, so we can apply equation (2.4) with $f(t) = 3t^2+2t+1$. Since $f'(t) = 6t+2$, we write $3t+1 = \tfrac{1}{2}(6t+2)$ before applying the formula. Thus we have

$$\begin{aligned}\int \frac{3t+1}{3t^2+2t+1}\, dt &= \tfrac{1}{2}\int \frac{6t+2}{3t^2+2t+1}\, dt \\ &= \tfrac{1}{2}\ln(3t^2+2t+1) + c,\end{aligned}$$

where c is an arbitrary constant.

(f) The factor $\cos x$ in the integrand is the derivative of $\sin x$, so we can apply equation (2.3) with $f(x) = \sin x$ and $n = 4$. Thus we have

$$\int \sin^4 x \cos x \, dx = \tfrac{1}{5}\sin^5 x + c,$$

where c is an arbitrary constant.

(g) The numerator e^{-2t} of the integrand is, except for a constant multiple, the derivative of the denominator $3e^{-2t}+2$. Also $3e^{-2t}+2 > 0$ for all t, so we can apply equation (2.4) with $f(t) = 3e^{-2t}+2$. Since $f'(t) = -6e^{-2t}$, we write $e^{-2t} = -\tfrac{1}{6}(-6e^{-2t})$ before applying the formula. Thus we have

$$\begin{aligned}\int \frac{e^{-2t}}{3e^{-2t}+2}\, dt &= -\tfrac{1}{6}\int \frac{-6e^{-2t}}{3e^{-2t}+2}\, dt \\ &= -\tfrac{1}{6}\ln(3e^{-2t}+2) + c,\end{aligned}$$

where c is an arbitrary constant.

(h) The factor x^2+1 in the integrand is, except for a constant multiple, the derivative of x^3+3x+2, so we can apply equation (2.3) with $f(x) = x^3+3x+2$ and $n = \tfrac{1}{2}$. Since $f'(x) = 3x^2+3$, we write $x^2+1 = \tfrac{1}{3}(3x^2+3)$ before applying the formula. Thus we have

$$\begin{aligned}&\int \sqrt{x^3+3x+2}\,(x^2+1)\, dx \\ &= \tfrac{1}{3}\int (x^3+3x+2)^{1/2}(3x^2+3)\, dx \\ &= \tfrac{2}{9}(x^3+3x+2)^{3/2} + c,\end{aligned}$$

where c is an arbitrary constant.

Solution 2.4

(a) Since $y = \cot x = \cos x/\sin x$, the Quotient Rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sin^2 x}\left(\sin x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sin x)\right) \\ &= \operatorname{cosec}^2 x\,(\sin x(-\sin x) - \cos x(\cos x)) \\ &= -\operatorname{cosec}^2 x\left(\sin^2 x + \cos^2 x\right) \\ &= -\operatorname{cosec}^2 x.\end{aligned}$$

(b) It follows from part (a) that

$$\begin{aligned}\int \operatorname{cosec}^2 x\, dx &= -\int (-\operatorname{cosec}^2 x)\, dx \\ &= -\cot x + c,\end{aligned}$$

where c is an arbitrary constant.

(c) It follows from the formula $\operatorname{cosec}^2 x = 1 + \cot^2 x$ that $\cot^2 x = \operatorname{cosec}^2 x - 1$. Using the result of part (b), we have

$$\begin{aligned}\int \cot^2 x\, dx &= \int (\operatorname{cosec}^2 x - 1)\, dx \\ &= -\cot x - x + c,\end{aligned}$$

where c is an arbitrary constant.

(d) Using the result of part (a), $\operatorname{cosec}^2 x$ is, except for a constant multiple, the derivative of $\cot x$. Hence we apply equation (2.3) with $f(x) = \cot x$ and $n = 2$. Since $f'(x) = -\operatorname{cosec}^2 x$, we write $\operatorname{cosec}^2 x = -(-\operatorname{cosec}^2 x)$ before applying the formula. Thus we have

$$\begin{aligned}\int \cot^2 x \operatorname{cosec}^2 x\, dx &= -\int \cot^2 x(-\operatorname{cosec}^2 x)\, dx \\ &= -\tfrac{1}{3}\cot^3 x + c,\end{aligned}$$

where c is an arbitrary constant.

Solution 2.5

(a) Starting from the given right-hand side, we have

$$\begin{aligned} x+\frac{x}{x^2-1} &= \frac{x(x^2-1)+x}{x^2-1} \\ &= \frac{x^3-x+x}{x^2-1} \\ &= \frac{x^3}{x^2-1}. \end{aligned}$$

Hence we can write

$$\begin{aligned} \int \frac{x^3}{x^2-1}\,dx &= \int \left(x+\frac{x}{x^2-1}\right)dx \\ &= \int x\,dx + \int \frac{x}{x^2-1}\,dx \\ &= \tfrac{1}{2}x^2 + \int \frac{x}{x^2-1}\,dx. \end{aligned}$$

The remaining integral can be found by applying equation (2.4) with $f(x)=x^2-1$, noting that $x^2-1>0$ for $x>1$. Since $f'(x)=2x$, we write $x=\frac{1}{2}(2x)$ before applying the formula. Thus we have

$$\begin{aligned} \int \frac{x}{x^2-1}\,dx &= \tfrac{1}{2}\int \frac{2x}{x^2-1}\,dx \\ &= \tfrac{1}{2}\ln(x^2-1)+c, \end{aligned}$$

where c is an arbitrary constant. It follows that

$$\int \frac{x^3}{x^2-1}\,dx = \tfrac{1}{2}x^2+\tfrac{1}{2}\ln(x^2-1)+c.$$

(b) On multiplying the top and bottom of the fraction in the integrand by e^{2t}, we have

$$\int \frac{1}{7+e^{-2t}}\,dt = \int \frac{e^{2t}}{7e^{2t}+1}\,dt.$$

On the right-hand side, the numerator of the integrand is, except for a constant multiple, the derivative of the denominator. Also $7e^{2t}+1>0$ for all t, so we can apply equation (2.4) with $f(t)=7e^{2t}+1$. Since $f'(t)=14e^{2t}$, we write $e^{2t}=\frac{1}{14}(14e^{2t})$ before applying the formula. Thus we have

$$\begin{aligned} \int \frac{1}{7+e^{-2t}}\,dt &= \tfrac{1}{14}\int \frac{14e^{2t}}{7e^{2t}+1}\,dt \\ &= \tfrac{1}{14}\ln(7e^{2t}+1)+c, \end{aligned}$$

where c is an arbitrary constant.

Solution 2.6

(a) We have

$$\begin{aligned} \int \frac{e^{2x}+e^{-2x}}{3e^x}\,dx &= \tfrac{1}{3}\int (e^x+e^{-3x})\,dx \\ &= \tfrac{1}{3}e^x - \tfrac{1}{9}e^{-3x}+c, \end{aligned}$$

where c is an arbitrary constant.

(b) The integrand is $\cot\theta=\cos\theta/\sin\theta$. Here the numerator $\cos\theta$ is the derivative of the denominator. Also $\sin\theta>0$ for $0<\theta<\pi$, so we can apply equation (2.4) with $f(\theta)=\sin\theta$. Thus we have

$$\begin{aligned} \int \cot\theta\,d\theta &= \int \frac{\cos\theta}{\sin\theta}\,d\theta \\ &= \ln(\sin\theta)+c, \end{aligned}$$

where c is an arbitrary constant.

(c) The integrand is $\sqrt{2-t}=(2-t)^{1/2}$, which is of the form $(f(t))^{1/2}$, where $f(t)=2-t$ and $f'(t)=-1$ is a constant. Hence we can apply equation (2.3) with $f(t)=2-t$ and $n=\frac{1}{2}$, after adjusting for the factor -1. Thus we have

$$\begin{aligned} \int \sqrt{2-t}\,dt &= -\int (2-t)^{1/2}(-1)\,dt \\ &= -\tfrac{2}{3}(2-t)^{3/2}+c, \end{aligned}$$

where c is an arbitrary constant.

(d) The factor $\sec^2 x$ in the integrand is the derivative of $\tan x$, so we can apply equation (2.3) with $f(x)=\tan x$ and $n=2$. Thus we have

$$\int \sec^2 x\tan^2 x\,dx = \tfrac{1}{3}\tan^3 x+c,$$

where c is an arbitrary constant.

(e) We have

$$\begin{aligned} \sec^4 x &= \sec^2 x\sec^2 x \\ &= \sec^2 x(1+\tan^2 x) \\ &= \sec^2 x+\sec^2 x\tan^2 x. \end{aligned}$$

Hence, using the result of part (d), we have

$$\begin{aligned} \int \sec^4 x\,dx &= \int (\sec^2 x+\sec^2 x\tan^2 x)\,dx \\ &= \tan x+\tfrac{1}{3}\tan^3 x+c, \end{aligned}$$

where c is an arbitrary constant.

(f) Multiplying the top and bottom of the fraction in the integrand by t gives

$$\int \frac{1}{t+1/t}\,dt = \int \frac{t}{t^2+1}\,dt.$$

On the right-hand side, the numerator is, except for a constant multiple, the derivative of the denominator. Also $t^2+1>0$ for all t, so we can apply equation (2.4) with $f(t)=t^2+1$. Since $f'(t)=2t$, we write $t=\frac{1}{2}(2t)$ before applying the formula. This gives

$$\begin{aligned} \int \frac{1}{t+1/t}\,dt &= \tfrac{1}{2}\int \frac{2t}{t^2+1}\,dt \\ &= \tfrac{1}{2}\ln(t^2+1)+c, \end{aligned}$$

where c is an arbitrary constant.

Solution 3.1

(a) Choose the s-axis to point up the slope, with $s = 0$ at the bottom of the hill, and set $t = 0$ at the moment when the car starts to climb. Then, using equation (3.6) of Chapter C2 Section 3, we have

$$v^2 - 2as = v_0^2 - 2as_0,$$

where $a = -1$, $v_0 = 20$ and $s_0 = 0$.

Thus, when $v = 0$, we obtain

$$-2(-1)s = 400; \quad \text{that is,} \quad s = 200.$$

Hence the car would come to rest after 200 metres.

(b) Using equation (3.6) again, with $s = 150$, we have

$$v^2 + 2 \times 150 = 400,$$

so that $v^2 = 100$ or $v = 10$. Hence the velocity of the car at the top of the hill is $10\,\mathrm{m\,s^{-1}}$.

(c) Now we choose an s-axis to point down the slope, with $s = 0$ at the top of the hill and $t = 0$ at the moment when the car starts to descend. Then, using equation (3.6) again, we have

$$v^2 - 2as = v_0^2 - 2as_0,$$

where $a = 0.5$, $v_0 = 10$ and $s_0 = 0$.

This gives $v^2 - s = 100$, and so $v = 20$ where

$$s = 400 - 100 = 300.$$

Hence the velocity of the car reaches $20\,\mathrm{m\,s^{-1}}$ again at 300 metres down the slope.

(d) To find the times for the ascent and descent, it is easiest to use equation (3.1) of Chapter C2 Section 3,

$$v = v_0 + at.$$

For the ascent, we have $v_0 = 20$, $a = -1$, and at the top of the hill $v = 10$, giving

$$10 = 20 - t; \quad \text{that is,} \quad t = 10.$$

Hence the ascent takes 10 seconds.

For the descent, we have $v_0 = 10$, $v = 20$ and $a = 0.5$, giving

$$20 = 10 + 0.5t; \quad \text{that is,} \quad t = 20.$$

Hence the descent takes 20 seconds.

The total time taken is therefore 30 seconds.

Solution 3.2

(a) We have $a = dv/dt = -2.5\cos(5t)$, so that

$$\begin{aligned} v &= \int (-2.5\cos(5t))\,dt \\ &= -\tfrac{1}{2}\sin(5t) + c, \end{aligned}$$

where c is an arbitrary constant. Since $v = 0$ when $t = 0$, we have

$$0 = 0 + c; \quad \text{that is,} \quad c = 0.$$

Hence the velocity is given by

$$v = -\tfrac{1}{2}\sin(5t).$$

(b) We have $v = ds/dt = -\frac{1}{2}\sin(5t)$, so that

$$\begin{aligned} s &= \int (-\tfrac{1}{2}\sin(5t))\,dt \\ &= \tfrac{1}{10}\cos(5t) + c, \end{aligned}$$

where c is an arbitrary constant. Since $s = 0.5$ when $t = 0$, we have

$$0.5 = 0.1 + c; \quad \text{that is,} \quad c = 0.4.$$

Hence the position of the particle below the ceiling is given by

$$s = 0.1\cos(5t) + 0.4.$$

(c) The highest point reached by the particle corresponds to the smallest value of s. Since $\cos(5t)$ takes values between -1 and 1, the smallest value of s is given by

$$s = -0.1 + 0.4 = 0.3,$$

and this occurs for the first time when $5t = \pi$; that is, when $t = \frac{1}{5}\pi \simeq 0.6$. Hence the highest point reached is 0.3 metres below the ceiling, and this occurs first at about 0.6 seconds after the particle is released.

(d) The maximum speed $|v|$ occurs when $|\sin(5t)| = 1$, and is therefore $\frac{1}{2} = 0.5$. Now $|\sin(5t)| = 1$ when $\cos(5t) = 0$, which gives $s = 0.4$. Hence the maximum speed of $0.5\,\mathrm{m\,s^{-1}}$ occurs where the particle is 0.4 metres below the ceiling.

Solution 4.1

(a) An integral of $3x^{1/2}$ is $2x^{3/2}$, so we have

$$\int_1^4 3\sqrt{x}\,dx = \left[2x^{3/2}\right]_1^4 = 2(8-1) = 14.$$

(b) For $x > 0$, an integral of $x^2 - 3x + 5/x$ is $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 5\ln x$, so

$$\begin{aligned}&\int_1^2 \left(x^2 - 3x + \frac{5}{x}\right) dx\\ &= \left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 5\ln x\right]_1^2\\ &= \tfrac{8}{3} - 6 + 5\ln 2 - \left(\tfrac{1}{3} - \tfrac{3}{2} + 5\ln 1\right)\\ &= -\tfrac{13}{6} + 5\ln 2 \simeq 1.299.\end{aligned}$$

(c) An integral of e^{2t} is $\frac{1}{2}e^{2t}$, so

$$\begin{aligned}\int_{-1}^1 e^{2t} dt &= \left[\tfrac{1}{2}e^{2t}\right]_{-1}^1\\ &= \tfrac{1}{2}(e^2 - e^{-2}) \simeq 3.627.\end{aligned}$$

(d) Using equation (2.4) of Chapter C2 Section 2, with $f(x) = 3x^2 + 1$, so that $f'(x) = 6x$, an integral of $x/(3x^2+1)$ is $\frac{1}{6}\ln(3x^2+1)$. Hence

$$\begin{aligned}\int_0^2 \frac{x}{3x^2+1}\,dx &= \left[\tfrac{1}{6}\ln(3x^2+1)\right]_0^2\\ &= \tfrac{1}{6}\ln 13 - \tfrac{1}{6}\ln 1\\ &= \tfrac{1}{6}\ln 13 \simeq 0.427.\end{aligned}$$

(e) An integral of $\cos(x/3)$ is $3\sin(x/3)$, so

$$\begin{aligned}\int_0^{\pi/2} \cos\left(\frac{x}{3}\right) dx &= \left[3\sin\left(\frac{x}{3}\right)\right]_0^{\pi/2}\\ &= 3\sin(\tfrac{1}{6}\pi) - 3\sin 0\\ &= \tfrac{3}{2} = 1.5.\end{aligned}$$

Solution 4.2

(a) The function $f(x) = 1 + x^2$ is positive for all x, so the integral

$$\int_{-1}^1 (1 + x^2)\,dx$$

represents the area under the graph of $1 + x^2$ from -1 to 1.

(b) The function $f(x) = \sin x$ takes negative values for $-\pi/2 \le x < 0$. Thus the integral

$$\int_{-\pi/2}^{\pi/2} \sin x\,dx$$

does not represent an area.

(c) The function $f(x) = 3 - x$ takes negative values for $x > 3$. Thus the integral

$$\int_0^5 (3 - x)\,dx$$

does not represent an area.

(d) The function $f(x) = e^{-2x}$ is positive for all x, so the integral

$$\int_{-1}^1 e^{-2x}\,dx$$

represents the area under the graph of e^{-2x} from -1 to 1.

Solution 4.3

(a) For $0 < t < 2$, we have $t > 0$, $t - 2 < 0$ and $t - 4 < -2 < 0$. Hence $f(t) = t(t-2)(t-4)$ is the product of one positive factor and two negative ones, so $f(t) > 0$.

It follows that the area required is

$$\begin{aligned}\int_0^2 t(t-2)(t-4)\,dt &= \int_0^2 (t^3 - 6t^2 + 8t)\,dt\\ &= \left[\tfrac{1}{4}t^4 - 2t^3 + 4t^2\right]_0^2\\ &= 4 - 16 + 16 - 0 = 4.\end{aligned}$$

(b) For $0 < x < \frac{1}{2}\pi$, we have $\sin x > 0$ and $\cos x > 0$, and so also $\sin x \cos x > 0$. Hence the area required is

$$\int_0^{\pi/2} \sin x \cos x\,dx.$$

From the solution to Exercise 2.2(a), an integral of $\sin x \cos x$ is $-\frac{1}{4}\cos(2x)$, so

$$\begin{aligned}\int_0^{\pi/2} \sin x \cos x\,dx &= \left[-\tfrac{1}{4}\cos(2x)\right]_0^{\pi/2}\\ &= -\tfrac{1}{4}\cos\pi + \tfrac{1}{4}\cos 0 = \tfrac{1}{2}.\end{aligned}$$

Alternatively, using the first solution to Exercise 2.3(c), we have

$$\begin{aligned}\int_0^{\pi/2} \sin x \cos x\,dx &= \left[\tfrac{1}{2}\sin^2 x\right]_0^{\pi/2}\\ &= \tfrac{1}{2}\sin^2(\tfrac{1}{2}\pi) - \tfrac{1}{2}\sin^2 0 = \tfrac{1}{2}.\end{aligned}$$

(c) (i) Using the Product Rule, the derivative of $(x+1)e^{-x}$ is

$$\begin{aligned}&\frac{d}{dx}(x+1)e^{-x} + (x+1)\frac{d}{dx}(e^{-x})\\ &= (1)e^{-x} + (x+1)(-e^{-x})\\ &= (1 - x - 1)e^{-x} = -xe^{-x},\end{aligned}$$

as required.

(ii) From the result of part (i), an integral of xe^{-x} is $-(x+1)e^{-x}$. Also $xe^{-x} > 0$ for $x > 0$, so the area required is

$$\begin{aligned}\int_0^2 xe^{-x} dx &= \left[-(x+1)e^{-x}\right]_0^2\\ &= -3e^{-2} - (-1e^0)\\ &= 1 - 3e^{-2} \simeq 0.594.\end{aligned}$$

Solutions for Chapter C3

Solution 1.1

(a) The differential equation is $dy/dx = x^4$. The derivative of the given function, $y = \frac{1}{5}x^5 + 1$, is

$$\frac{d}{dx}\left(\tfrac{1}{5}x^5 + 1\right) = \tfrac{1}{5}(5x^4) = x^4,$$

which is the same expression as the right-hand side of the differential equation, as required.

(b) The differential equation is $dy/dx = 2\sin x \cos x$. In order to differentiate the given function, $y = \sin^2 x$, we apply the Composite (Chain) Rule. The function can be written as $y = u^2$, where $u = \sin x$, for which

$$\frac{dy}{du} = 2u \quad \text{and} \quad \frac{du}{dx} = \cos x.$$

Hence we have

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} &= (2u)(\cos x)\\ &= 2\sin x \cos x,\end{aligned}$$

which is the same expression as the right-hand side of the differential equation, as required.

(c) The differential equation is $dy/dx = 3y - 3x + 1$. Differentiating the given function, $y = 5e^{3x} + x$, we have

$$\frac{dy}{dx} = 15e^{3x} + 1.$$

Now we substitute $y = 5e^{3x} + x$ into the right-hand side of the differential equation, to obtain

$$\begin{aligned}3y - 3x + 1 &= 3(5e^{3x} + x) - 3x + 1\\ &= 15e^{3x} + 1.\end{aligned}$$

This is the same expression as was obtained above for the derivative, as required.

(d) The differential equation is $dy/dx = -2xy$. We differentiate the given function $y = e^{-x^2}$ using the Composite (Chain) Rule. The function can be written as $y = e^u$, where $u = -x^2$, for which

$$\frac{dy}{du} = e^u \quad \text{and} \quad \frac{du}{dx} = -2x.$$

Hence we have

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} &= (e^u)(-2x)\\ &= -2xe^{-x^2}.\end{aligned}$$

Substituting $y = e^{-x^2}$ into the right-hand side of the differential equation, we obtain

$$-2xy = -2xe^{-x^2},$$

which is the same expression as was obtained above for the derivative, as required.

(e) The differential equation is $dy/dx = 1 + y^2$. The derivative of the given function, $y = \tan x$, is

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Now we substitute $y = \tan x$ into the right-hand side of the differential equation, to obtain

$$1 + y^2 = 1 + \tan^2 x = \sec^2 x,$$

which is the same expression as was obtained above for the derivative, as required.

(f) The differential equation is $dy/dx = (x + y)/x$. In order to differentiate the given function, $y = x\ln x$, we apply the Product Rule. The function can be written as $y = uv$, where $u = x$ and $v = \ln x$, for which

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = \frac{1}{x}.$$

Hence we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx}\\ &= (1)(\ln x) + (x)\left(\frac{1}{x}\right) = 1 + \ln x.\end{aligned}$$

Now we substitute $y = x\ln x$ into the right-hand side of the differential equation, to obtain

$$\frac{x + y}{x} = \frac{x + x\ln x}{x} = 1 + \ln x,$$

which is the same expression as was obtained above for the derivative, as required.

Solution 1.2

In each case, c is an arbitrary constant.

(a) The general solution of $dy/dx = x^2 - 3x + 1$ is

$$\begin{aligned}y &= \int (x^2 - 3x + 1)\,dx\\ &= \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + x + c.\end{aligned}$$

(b) The general solution of $dy/dx = \sin(2x) + \cos(2x)$ is

$$\begin{aligned}y &= \int (\sin(2x) + \cos(2x))\,dx\\ &= -\tfrac{1}{2}\cos(2x) + \tfrac{1}{2}\sin(2x) + c.\end{aligned}$$

(c) The general solution of $dv/dt = e^{-3t}$ is

$$v = \int e^{-3t}dt = -\tfrac{1}{3}e^{-3t} + c.$$

(d) The general solution of $dy/dx = x^3 - 1/x + 3\sqrt{x}$ is

$$\begin{aligned}y &= \int \left(x^3 - \frac{1}{x} + 3\sqrt{x}\right)dx\\ &= \int \left(x^3 - \frac{1}{x} + 3x^{1/2}\right)dx\\ &= \tfrac{1}{4}x^4 - \ln x + 3\left(\tfrac{2}{3}x^{3/2}\right) + c\\ &= \tfrac{1}{4}x^4 - \ln x + 2x\sqrt{x} + c.\end{aligned}$$

Solution 1.3

(a) The general solution of $dy/dx = 2 - 3\sin(4x)$ is

$$\begin{aligned} y &= \int (2 - 3\sin(4x))\,dx \\ &= 2x + \tfrac{3}{4}\cos(4x) + c, \end{aligned}$$

where c is an arbitrary constant. Using the initial condition, $y = \frac{1}{4}$ when $x = 0$, we have

$$\tfrac{1}{4} = 0 + \tfrac{3}{4} + c.$$

Hence $c = -\frac{1}{2}$, and the solution of the initial-value problem is

$$y = 2x + \tfrac{3}{4}\cos(4x) - \tfrac{1}{2}.$$

(b) The general solution of $du/dt = 1/\sqrt{t} - 1/t$, for $t > 0$, is

$$\begin{aligned} u &= \int \left(\frac{1}{\sqrt{t}} - \frac{1}{t}\right) dt \\ &= \int \left(t^{-1/2} - \frac{1}{t}\right) dt \\ &= 2t^{1/2} - \ln t + c, \end{aligned}$$

where c is an arbitrary constant. Using the initial condition, $u = 3$ when $t = 1$, we have

$$3 = 2 - 0 + c.$$

Hence $c = 1$, and the solution of the initial-value problem is

$$u = 2\sqrt{t} - \ln t + 1.$$

(c) The general solution of $dv/dr = 4\pi r^2$ is

$$\begin{aligned} v &= \int 4\pi r^2\,dr \\ &= \tfrac{4}{3}\pi r^3 + c, \end{aligned}$$

where c is an arbitrary constant. Using the initial condition, $v = 0$ when $r = 0$, we have

$$0 = 0 + c.$$

Hence $c = 0$, and the solution of the initial-value problem is

$$v = \tfrac{4}{3}\pi r^3.$$

(d) The general solution of $ds/dt = \cos\left(\frac{1}{2}t\right)$ is

$$\begin{aligned} s &= \int \cos\left(\tfrac{1}{2}t\right) dt \\ &= 2\sin\left(\tfrac{1}{2}t\right) + c, \end{aligned}$$

where c is an arbitrary constant. Using the initial condition, $s = 0$ when $t = \pi$, we have

$$0 = 2 + c.$$

Hence $c = -2$, and the solution of the initial-value problem is

$$s = 2\sin(\tfrac{1}{2}t) - 2.$$

(e) The general solution of $dy/dt = e^t + e^{2t}$ is

$$\begin{aligned} y &= \int (e^t + e^{2t})\,dt \\ &= e^t + \tfrac{1}{2}e^{2t} + c, \end{aligned}$$

where c is an arbitrary constant. Using the initial condition, $y = 2$ when $t = 0$, we have

$$2 = 1 + \tfrac{1}{2} + c.$$

Hence $c = \frac{1}{2}$, and the solution of the initial-value problem is

$$y = e^t + \tfrac{1}{2}e^{2t} + \tfrac{1}{2}.$$

Solution 1.4

(a) The general solution of $ds/dt = 120e^{-0.1t} - 100$ is

$$\begin{aligned} s &= \int (120e^{-0.1t} - 100)\,dt \\ &= -1200e^{-0.1t} - 100t + c, \end{aligned}$$

where c is an arbitrary constant.

(b) Using the initial condition, $s = 0$ when $t = 0$, we have

$$0 = -1200 + c.$$

Hence $c = 1200$, and the height of the ball at time t is given by

$$s = -1200e^{-0.1t} - 100t + 1200;$$

that is,

$$s = 1200(1 - e^{-0.1t}) - 100t.$$

(c) Now $ds/dt = 0$ when $120e^{-0.1t} - 100 = 0$. Hence

$$e^{-0.1t} = \tfrac{100}{120} = \tfrac{5}{6},$$

from which we obtain

$$-0.1t = \ln\left(\tfrac{5}{6}\right) = -\ln\left(\tfrac{6}{5}\right);$$

that is,

$$t = 10\ln(\tfrac{6}{5}).$$

The ball reaches its maximum height when its velocity ds/dt is zero, and as just shown, this occurs at time $t = 10\ln(\frac{6}{5})$. From part (b), the corresponding height is

$$\begin{aligned} s &= 1200(1 - \tfrac{5}{6}) - 100 \times 10\ln(\tfrac{6}{5}) \\ &= 200 - 1000\ln(\tfrac{6}{5}) \simeq 17.68. \end{aligned}$$

Hence the maximum height reached is 17.68 metres (to the nearest cm).

Solution 2.1

(a) The given equation is of the form $H(y) = F(x)$, with $H(y) = \cos y$ and $F(x) = x$. It follows from result (2.3) of Chapter C3 Section 2 that

$$\frac{d}{dy}(\cos y)\frac{dy}{dx} = \frac{d}{dx}(x)$$

which leads to

$$-\sin y\frac{dy}{dx} = 1.$$

Dividing through by $-\sin y$ (for $0 < y < \pi$) gives

$$\frac{dy}{dx} = -\frac{1}{\sin y}.$$

Using the identity $\sin^2 y + \cos^2 y = 1$, we have

$$\begin{aligned}\sin y &= \pm\sqrt{1-\cos^2 y}\\ &= \pm\sqrt{1-x^2}.\end{aligned}$$

But $\sin y > 0$ for $0 < y < \pi$, and hence

$$\sin y = \sqrt{1-x^2},$$

giving

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \quad (0 < y < \pi),$$

as required.

(b) Using the definition of arccos from Chapter A3 Section 4, the equation $\cos y = x$ for $0 < y < \pi$ is equivalent to

$$y = \arccos x, \quad \text{where } -1 < x < 1.$$

Hence, from part (a), we have

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1).$$

Solution 2.2

(a) The given equation is of the form $H(y) = F(x)$, with $H(y) = y + e^y$ and $F(x) = 2x - e^x + 2$. It follows from result (2.3) of Chapter C3 Section 2 that

$$\frac{d}{dy}(y+e^y)\frac{dy}{dx} = \frac{d}{dx}(2x - e^x + 2),$$

which leads to

$$(1+e^y)\frac{dy}{dx} = 2 - e^x.$$

Dividing through by $1 + e^y$ gives

$$\frac{dy}{dx} = \frac{2-e^x}{1+e^y},$$

as required.

(b) At $(0, 0)$, the slope of the solution curve is

$$\frac{dy}{dx} = \frac{2-1}{1+1} = \tfrac{1}{2}.$$

Solution 2.3

(a) Comparing with result (2.4) of Chapter C3 Section 2, we have $h(y) = 1/(2y)$ and $f(x) = -3x^2$. Hence the general solution of this differential equation is given by

$$\int \frac{1}{2y}\,dy = \int (-3x^2)\,dx;$$

that is, by $\frac{1}{2}\ln y = -x^3 + c$, where c is an arbitrary constant.

(b) The explicit form of the general solution is obtained by making y the subject of the equation. In this case, we multiply by 2 and then take the exponential of both sides of the equation, which gives

$$y = e^{(-2x^3+2c)} \quad \text{or} \quad y = Ae^{-2x^3},$$

where $A = e^{2c}$ is a positive, but otherwise arbitrary, constant.

(c) Using the initial condition, $y = 4$ when $x = 0$, we have $4 = A$, so the required particular solution is $y = 4e^{-2x^3}$.

Solution 2.4

(a) The differential equation $dy/dx = 2xy^2$ has a right-hand side of the form $f(x)g(y)$, where $f(x) = 2x$ and $g(y) = y^2$. Dividing both sides of the differential equation by $g(y)$ gives

$$\frac{1}{y^2}\frac{dy}{dx} = 2x.$$

From this we obtain, using result (2.4) of Chapter C3 Section 2,

$$\int \frac{1}{y^2}\,dy = \int 2x\,dx.$$

On integrating, we find that

$$-\frac{1}{y} = x^2 + c,$$

where c is an arbitrary constant.

This implicit form of the general solution can be manipulated to make y the subject, giving the explicit form

$$y = -\frac{1}{x^2+c}.$$

The initial condition is $y = 1$ when $x = 0$. Putting $x = 0$ and $y = 1$ into the general solution, we have $1 = -1/c$. Hence $c = -1$, and the solution of the initial-value problem is

$$y = -\frac{1}{x^2-1} \quad \text{or} \quad y = \frac{1}{1-x^2}.$$

(Since we were given $y > 0$, we must have $-1 < x < 1$ for this solution to be valid.)

In the remaining parts of this solution, we use the shortcut described in the text before Activity 2.4 of Chapter C3 Section 2.

(b) The differential equation $dx/dt = 2(x-2)/t$ has a right-hand side of the form $f(t)g(x)$, where $f(t) = 2/t$ and $g(x) = x - 2$. Separating the variables in the differential equation, we obtain

$$\int \frac{1}{x-2}\,dx = \int \frac{2}{t}\,dt.$$

For the left-hand side, we use equation (2.4) of Chapter C2 Section 2. Since $g(x) = x - 2 > 0$, and $g'(x) = 1$, this gives

$$\int \frac{1}{x-2}\,dx = \ln(x-2) + \text{constant}.$$

Hence, on carrying out both of the integrations above, we have

$$\ln(x-2) = 2\ln t + c,$$

where c is an arbitrary constant.

Rearranging this to obtain the general solution in explicit form, we have

$$\ln(x-2) = \ln(t^2) + c,$$

so that

$$x - 2 = e^{\ln(t^2)+c} = e^c t^2,$$

and hence $x = At^2 + 2$, where $A = e^c$ is a positive, but otherwise arbitrary, constant.

Using the initial condition, $x = 4$ when $t = 1$, we have $4 = A + 2$. Hence $A = 2$, and the solution of the initial-value problem is

$$x = 2t^2 + 2.$$

(c) The differential equation $dy/dx = e^{-y}$ has a right-hand side of the form $f(x)g(y)$, where $f(x) = 1$ and $g(y) = e^{-y}$. Separating the variables in the differential equation, we obtain

$$\int e^y\,dy = \int 1\,dx.$$

On integrating, we have

$$e^y = x + c,$$

where c is an arbitrary constant.

To obtain the explicit form of the general solution, we take the natural logarithm of both sides, which gives

$$y = \ln(x + c).$$

Using the initial condition, $y = 1$ when $x = 0$, we have $1 = \ln c$. Hence $c = e$, and the solution of the initial-value problem is

$$y = \ln(x + e).$$

(We must have $x > -e$ for this solution to be valid.)

(d) The differential equation $du/dr = 2r(1 + u^2)$ has a right-hand side of the form $f(r)g(u)$, where $f(r) = 2r$ and $g(u) = 1 + u^2$. Separating the variables in the differential equation, we obtain

$$\int \frac{1}{1+u^2}\,du = \int 2r\,dr.$$

On integrating, and applying the first result in the Comment for Activity 2.2 for the left-hand side, we have

$$\arctan u = r^2 + c,$$

where c is an arbitrary constant. Using the initial condition, $u = 1$ when $r = 0$, we have $\arctan 1 = c$. Hence $c = \frac{1}{4}\pi$, and the solution of the initial-value problem is

$$\arctan u = r^2 + \tfrac{1}{4}\pi.$$

In explicit form, this is

$$u = \tan\left(r^2 + \tfrac{1}{4}\pi\right).$$

Solution 2.5

(a) The method of separation of variables can be applied, since the right-hand side is of the form $f(x)g(y)$ with $f(x) = 3$ and $g(y) = \sec y$. Separating the variables, we obtain

$$\int \frac{1}{\sec y}\,dy = \int 3\,dx.$$

Since $1/\sec y = \cos y$, integration gives

$$\sin y = 3x + c,$$

where c is an arbitrary constant.

Rewriting this in explicit form, for $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$, we have

$$y = \arcsin(3x + c).$$

(b) The method of direct integration can be applied. We have

$$y = \int 3e^{2x}\,dx$$
$$= \tfrac{3}{2}e^{2x} + c,$$

where c is an arbitrary constant.

(c) Neither method can be applied here, as the equation is not of an appropriate form.

(d) The method of separation of variables can be applied, since the right-hand side, $u + ut = u(1 + t)$, is of the form $f(t)g(u)$ with $f(t) = 1 + t$ and $g(u) = u$. Separating the variables, we obtain

$$\int \frac{1}{u}\,du = \int (1 + t)\,dt.$$

On integrating, we have

$$\ln u = t + \tfrac{1}{2}t^2 + c,$$

where c is an arbitrary constant.

In explicit form, the general solution is

$$u = e^{t+(t^2/2)+c} \quad \text{or} \quad u = Ae^{t+(t^2/2)},$$

where $A = e^c$ is a positive, but otherwise arbitrary, constant.

(e) Neither method can be applied here, as the equation is not of an appropriate form.

Solution 2.6

(a) The differential equation is $dv/dt = -k(v + g/k)$, and the right-hand side is of the form $f(t)q(v)$, with $f(t) = -k$ and $q(v) = v + g/k$. Separating the variables, we have

$$\int \frac{1}{v + g/k}\, dv = \int (-k)\, dt.$$

For the left-hand side, we use equation (2.4) of Chapter C2 Section 2. Since $q(v) = v + g/k > 0$, and $q'(v) = 1$, this gives

$$\int \frac{1}{v + g/k}\, dv = \ln\left(v + \frac{g}{k}\right) + \text{constant}.$$

Hence, on carrying out both of the integrations above, we obtain

$$\ln\left(v + \frac{g}{k}\right) = -kt + c,$$

where c is an arbitrary constant.

Rearranging this to find the general solution in explicit form, we have

$$v + \frac{g}{k} = e^{-kt+c},$$

so that

$$v = Ae^{-kt} - \frac{g}{k},$$

where $A = e^c$ is a positive, but otherwise arbitrary, constant.

(b) Using the initial condition, $v = v_0$ when $t = 0$, we have

$$v_0 = A - \frac{g}{k}.$$

Hence $A = v_0 + g/k$, so the required particular solution is

$$v = \left(v_0 + \frac{g}{k}\right) e^{-kt} - \frac{g}{k}.$$

(c) Taking $v_0 = 20$, $g = 10$ and $k = 0.1$ gives

$$v = 120e^{-0.1t} - 100,$$

which is the required equation.

Solution 3.1

(a) Using equation (3.5) of Chapter C3 Section 3, the decay constant k corresponding to a half-life $T = 5.27$ years is

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{5.27} = 0.132 \text{ year}^{-1} \text{ (to 3 s.f.)}.$$

(b) Hence, from equation (3.4) of Chapter C3 Section 3, the proportion of a sample remaining after 8 years is

$$\frac{m(t+8)}{m(t)} = e^{-0.132\times 8} = 0.349 \text{ (to 3 s.f.)}.$$

So about 35% of the original amount of cobalt-60 remains after 8 years.

(c) The proportion remaining will be 10% of the original amount after a time T years, where

$$\frac{m(t+T)}{m(t)} = \tfrac{10}{100} = e^{-kT}.$$

Taking logarithms, we obtain

$$-kT = \ln(0.1),$$

whose solution is

$$T = -\frac{\ln(0.1)}{k} = \frac{\ln 10}{0.132} = 17.5 \text{ (to 3 s.f.)}.$$

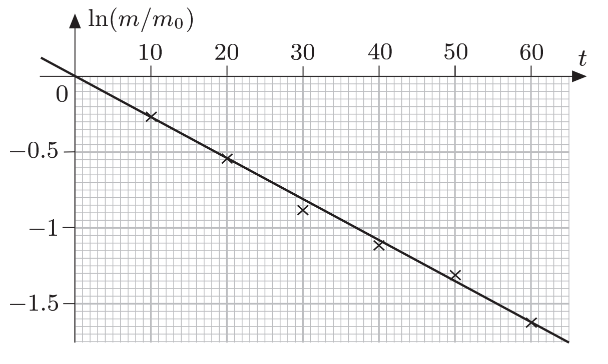
Hence it will be about $17\frac{1}{2}$ years before the proportion of cobalt-60 remaining is 10% of the original amount.

Solution 3.2

(a) The completed table of data is given below.

t (days)	10	20	30	40	50	60
m/m_0	0.76	0.58	0.42	0.33	0.27	0.20
$\ln(m/m_0)$	−0.27	−0.54	−0.87	−1.11	−1.31	−1.61

(b) The graph of $\ln(m/m_0)$ against t for the data is shown below. We have added a straight line (chosen by eye) through the origin which fits the data points well. (This good fit confirms that the exponential model, $m = m_0e^{-kt}$, gives a satisfactory description of the radioactive decay of strontium-82.)



(c) The straight line passes through the origin and through the point $(60, -1.62)$, and so its slope is

$$-k = \frac{-1.62}{60} = -0.027.$$

Hence the decay constant for strontium-82 is $k = 0.027\ \text{day}^{-1}$. From equation (3.5) of Chapter C3 Section 3, the corresponding half-life is

$$T = \frac{\ln 2}{k} = \frac{\ln 2}{0.027} = 25.7 \text{ days (to 3 s.f.)}.$$

(d) After 90 days, the proportion of the sample remaining is estimated to be

$$e^{-90k} = e^{-90\times 0.027} = 0.0880 \text{ (to 3 s.f.)}.$$

So, according to the model, about 9% of the original amount of strontium-82 remains after 90 days.

Solution 3.3

(a) The initial population size is $P_0 = 8 \times 10^6$, and the proportionate growth rate is $K = 0.02\ \text{year}^{-1}$. Hence, from equation (3.7) of Chapter C3 Section 3, the population size P after time t years is

$$P = P_0 e^{Kt} = 8 \times 10^6 e^{0.02t}.$$

After 10 years the population size is

$$P = 8 \times 10^6 e^{0.2} = 9.77 \times 10^6 \text{ (to 3 s.f.)},$$

whereas after 40 years the population size is

$$P = 8 \times 10^6 e^{0.8} = 17.8 \times 10^6 \text{ (to 3 s.f.)}.$$

So, to 3 significant figures, the population size after 10 years is 9.77 million, and after 40 years it is 17.8 million.

(b) The population size reaches 80 million (80×10^6) at the time t years given by

$$80 = 8e^{0.02t}; \quad \text{that is,} \quad 10 = e^{0.02t}.$$

Taking the logarithm of each side, we obtain $0.02t = \ln 10$; that is,

$$t = \frac{\ln 10}{0.02} = 115 \text{ (to 3 s.f.)}.$$

Hence it takes about 115 years for the population size to reach 80 million.

(c) The doubling time of the population is given by

$$T = \frac{\ln 2}{K}$$

(see Activity 3.6(a) of Chapter C3 Section 3). For this population we have $K = 0.02$, so the doubling time is

$$T = \frac{\ln 2}{0.02} = 34.7 \text{ (to 3 s.f.)}.$$

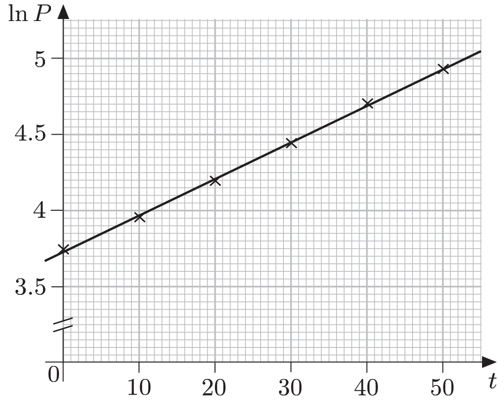
Hence the doubling time is just under 35 years.

Solution 3.4

(a) The table below gives the values of the time t (in years since 1950), the population size P (in millions) and $\ln P$, to 3 significant figures.

t	0	10	20	30	40	50
P	42	52	66	85	110	137
$\ln P$	3.74	3.95	4.19	4.44	4.70	4.92

The corresponding log-linear plot is shown below.



The data points are fitted reasonably well by the straight line drawn. This confirms that the exponential model is appropriate to describe the Bangladeshi population for the years 1950–2000.

(b) The intercept on the $(\ln P)$-axis is 3.72. This is the value of $\ln P_0$ for the model, which gives

$$P_0 = e^{3.72} = 41.3 \text{ (to 3 s.f.)}.$$

As well as the point $(0, 3.72)$, another point on the straight line is $(50, 4.92)$. The slope of the straight line is therefore

$$\frac{4.92 - 3.72}{50 - 0} = 0.024.$$

So our estimate for the proportionate growth rate K is $0.024\ \text{year}^{-1}$, and for the initial population P_0 the estimate is 41.3 million.

(c) Using the estimates obtained for K and P_0, the exponential model for the Bangladeshi population (1950–2000) is

$$P = 41.3e^{0.024t}.$$

The year 2010 corresponds to $t = 60$, so the estimate from the model for the population in that year is

$$P(60) = 41.3e^{0.024\times 60} = 174 \text{ (to 3 s.f.)}.$$

Hence our estimate for the population of Bangladesh in 2010 is 174 million.

Solution 3.5

(a) Since $y = \theta - \theta_a$, and θ_a is constant,

$$\frac{dy}{dt} = \frac{d\theta}{dt}.$$

Hence equation (1) can be written as

$$\frac{dy}{dt} = -ky.$$

(b) From result (3.9), with $K = -k$, the general solution of this differential equation is

$$y = Ae^{-kt},$$

where A is an arbitrary constant. Hence the general solution of equation (1), for $\theta = y + \theta_a$, is

$$\theta = Ae^{-kt} + \theta_a.$$

(c) Using the initial condition, $\theta = \theta_0$ when $t = 0$, we have

$$\theta_0 = A + \theta_a.$$

Hence $A = \theta_0 - \theta_a$, and the required particular solution is

$$\theta = (\theta_0 - \theta_a)e^{-kt} + \theta_a.$$

(d) Taking $\theta_a = 20$ and $\theta_0 = 90$, the particular solution becomes

$$\theta = 70e^{-kt} + 20.$$

Since t is measured in seconds, we have $\theta = 60$ when $t = 300$, so that

$$60 = 70e^{-300k} + 20; \quad \text{that is,} \quad e^{-300k} = \tfrac{4}{7}.$$

Taking logarithms, we obtain

$$-300k = \ln\left(\tfrac{4}{7}\right),$$

so that

$$k = \tfrac{1}{300}\ln\left(\tfrac{7}{4}\right) = 0.001\,87 \text{ (to 3 s.f.)}.$$

Hence the temperature of the object is given by

$$\theta = 70e^{-0.001\,87t} + 20.$$

After a further 5 minutes (that is, 10 minutes from the start), we have $t = 600$, so the temperature is then given by

$$\theta = 70e^{-0.001\,87\times 600} + 20 = 42.9.$$

Hence the temperature of the object after a further 5 minutes is estimated to be about 43°C.

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